# Control of Nonlinear Physiological Systems via LPV Framework

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***Abstract*—We introduce a controller design methodology for nonlinear systems via complementary Linear Parameter Varying (LPV) controller and observer structures. The recently developed method is able to control physiological systems even with complex nonlinearities – without using Linear Matrix Inequalities (LMI) or other techniques requiring iterations. The developed method is based on the classical state feedback theorems, matrix similarity theorems and supplementary controller and observer structure which efficiently uses the mathematical properties of the parameter space of the LPV system. The main benefits of the proposed method is that the controller design does not require mathematical tools needing iteration thus high computational capacity. We used a nonlinear compartmental model in order to demonstrate the application of the method. The results showed that the developed complementary LPV controller and observer structures perform well on both the LPV systems and the original nonlinear system as well.**

## I. Introduction

The control of physiological systems is challenging because most of them are modeled with nonlinear time-varying differential equations. Especially in the case of the control of different parameters in living organisms, where several related information are not known due to the lack of measurements [1].

In order to deal with these unfavorable circumstances, different controller design techniques can be used – mostly based on the Lyapunov's method [2]. However, most of these methods – for example Model Predictive Control-, LMI-, Fractional order-based controller design methods – require complex calculations and high computational capacity. Moreover, the adaptive methods become more and more important due to the lack of information regarding the system to be controlled. The recently developed Robust Fixed Point Transformation (RFPT)-based controller design [3] framework is able to provide robust and adaptive control solution based only on rough approximation of the real system [4]. LPV-based methods can also be useful in this regard. The main advantage of LPV-based framework is that it is able to "hide" nonlinearities of the system to be controlled and linear controller design approaches can be applied [5]. Although, usually the LPV methods are combined with Lyapunov's laws through LMI frameworks.

We demonstrate the usability of a complementary LPV controller and observer design approach. The method exploits the mathematical properties of the abstract parameter space of the LPV systems and the matrix similarity theorems [6], [7]. The paper is structured in the following way: first, we introduce the novel complementary LPV controller and observer scheme and the ideas behind; second, we demonstrate the developed method on an example nonlinear compartmental model; finally, we present the results of the simulations.

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## II. Overview of the Controller Design Method

We provide a short overview about the LPV-based controller designing method used in this paper. The method was recently developed in [6], [8], [9].

In general, the procedure does allow the controller design for the nonlinear system given by its state-space representation via LPV framework by combining the mathematical properties of the LPV parameter space (PS), matrix similarity theorems and classical state feedback control. Moreover, it does not require highly complex Linear Matrix Inequality (LMI)-based procedures and high computational capacity. The method can be used to design not just the complementary controller, but also complementary observer-structures as well.

### A. Used LPV systems in general

Generally the state-space representation of an LPV system is the following:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{D}(\mathbf{p}(t))\mathbf{u}(t)\end{aligned}, \tag{1}$$

where the state vector is $\mathbf{x}(t) \in \mathbb{R}^n$, the output vector is $\mathbf{y}(t) \in \mathbb{R}^k$, the input vector is $\mathbf{u}(t) \in \mathbb{R}^m$. $\mathbf{A}(\mathbf{p}(t)) \in \mathbb{R}^{n\times n}$ is the system matrix, $\mathbf{B}(\mathbf{p}(t)) \in \mathbb{R}^{n\times m}$ is the input matrix, $\mathbf{C}(\mathbf{p}(t)) \in \mathbb{R}^{k\times n}$ is the output matrix and $\mathbf{D}(\mathbf{p}(t)) \in \mathbb{R}^{k\times m}$ is the forward matrix.

The unified form of (1) is denoted by

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix}\mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t))\\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t))\end{pmatrix}, \tag{2}$$

where $\mathbf{S}(\mathbf{p}(t)) \in \mathbb{R}^{(n+k)\times(n+m)}$ is the unified system matrix.

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Usually, the compact form of a general LPV system from (1) can be written as:

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \mathbf{S}(\mathbf{p}(t))\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} \ . \qquad (3)$$

The affine form of an LPV system is

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{A}_0 + \sum_{i=1}^{q} p_i(t)\mathbf{A}_i & \mathbf{B}_0 + \sum_{i=1}^{q} p_i(t)\mathbf{B}_i \\ \mathbf{C}_0 + \sum_{i=1}^{q} p_i(t)\mathbf{C}_i & \mathbf{D}_0 + \sum_{i=1}^{q} p_i(t)\mathbf{D}_i \end{pmatrix}, \qquad (4)$$

where $\mathbf{A}_0, \mathbf{B}_0, \mathbf{C}_0$ and $\mathbf{D}_0$ are permanent terms that do not depend on the parameter vector. In the case of the affine form the $\mathbf{S}(\mathbf{p}(t))$ of (3) is equal to (4).

The most important properties from the used controller design method point of view are related to the PS of the given LPV system. The PS is determined by the parameter vector $\mathbf{p}(t) \in \mathbb{R}^q$, which occurs in (1)-(4). The $\mathbf{p}(t)$ is a bounded varying vector that consists of the so-called scheduling parameters. These can be constant ($p = \{p \in \mathbb{R}, p_{min} \le p \le p_{max}\}$) or depend on some variables ($p(x) = \{p(x) \in \mathbb{R}, p_{min} \le p(x) \le p_{max}\}$) of the original nonlinear model and determine particular properties of the system. The $\mathbf{p}(t)$ can contain state variables as well – in this case not just the model parameters, but fundamental properties of the model can vary over time [6]. The $\mathbf{p}(t)$ generates the PS, which is a $q$-dimensional real vector space $\mathbb{R}^q$, in which each dimension represents a scheduling parameter. The last missing component required for the design technique is the Parameter Box (PB), which is a particular part of the PS. The PB is a $q$-dimensional simplex inside the $\mathbb{R}^q$ determined by the minimum $p_{i,min}$ and maximum $p_{i,max}$ values of the scheduling parameters $p_i(t)$. In general, the PB means the region of interest inside the PS. The main goal during the selection of the scheduling parameters is to select the terms causing nonlinearity – in this way the nonlinearities can be hidden and linear controller design methods can be applied [5]. If the terms causing nonlinearity are selected and embedded into the $\mathbf{p}(t)$ vector, the remaining terms of the system are constant.

### B. Complementary LPV controller and observer structure

A nonlinear time-varying system can be represented in LPV form with the system matrix $\mathbf{S}(\mathbf{p}(t))$, where the terms causing nonlinearity are embedded into the vector $\mathbf{p}(t)$. Assume that the parameter vector is the fixed vector $\mathbf{p}_{fixed}$. In this case $\mathbf{S}(\mathbf{p}_{fixed})$ becomes a Linear Time Invariant (LTI) system. An LPV system $\mathbf{S}(\mathbf{p}(t))$ can be described with infinite underlying LTI systems. In our previous research we proved that it is possible to compare underlying LTI systems only via their actual parameter vectors based on a norm criteria. The basis of the comparison was predefined for 2-norm based difference of the corresponding parameter vectors in the PS [6], [8], [9]. The difference between two underlying LTI systems ($\mathbf{S}(\mathbf{p}_i)$ and $\mathbf{S}(\mathbf{p}_j)$) in the PS can be determined by $e := \|\mathbf{p}_i - \mathbf{p}_j\|_2$ [6]. This difference can be extended: it is possible to use it to measure the difference of a given LTI reference system $\mathbf{S}(\mathbf{p}_{ref})$ (which is an underlying LTI system as well) and the current LPV system $\mathbf{S}(\mathbf{p}(t))$. In this case, the difference of the systems (the difference of the parameter vectors) becomes time varying, i.e. $e(t) := \|\mathbf{p}_{fixed} - \mathbf{p}_{var}(t)\|_2$ [6].

In our previous study we demonstrated that by using $e(t)$ to describe the difference between $\mathbf{S}(\mathbf{p}_{ref})$ and $\mathbf{S}(\mathbf{p}(t))$ it is possible to realize such kind of complementary controller structure which is able to change the parameters of the controller based on the information gathered from $e(t)$ [6], [8], [9]. The underlying idea is that the matrix similarity theorems [7] do allow us to prescribe strict eigenvalue requirements against the LPV systems controlled by the complementary LPV controller. Based on these prescriptions, it is possible to realize a controller structure that enforces the controlled LPV system to behave as a given reference system – more precisely, the complementary controller structure enforces the controlled LPV system to mimic the reference LTI system. In details, the embedded matrix similarity theorems provide that the eigenvalues of the controlled LPV system (in closed-loop) are the same as the controlled reference LTI system. These properties can be reached by using state-feedback theorems [7], [9].

Suppose that we have designed a state-feedback controller for the reference system described by the control law $\mathbf{u}(t) = -\mathbf{K}_{ref}\mathbf{x}(t)$. Then the system matrix of the closed-loop reference system is $\mathbf{A}_{ref} - \mathbf{B}_{ref}\mathbf{K}_{ref}$ whose eigenvalues define the dynamics of the reference system. We want to ensure that the nonlinear system described by an LPV model has the same dynamics, so we want to guarantee that the eigenvalues of $\mathbf{A}_{ref} - \mathbf{B}\mathbf{K}_{ref}$ are equal to the eigenvalues of $\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(t)e(t))$ where we applied the control law $\mathbf{u}(t) = -(\mathbf{K}_{ref} + \mathbf{K}(t)e(t))\mathbf{x}(t)$.

Assume that $\mathbf{A}_{ref} - \mathbf{B}\mathbf{K}_{ref} \sim \mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(t)e(t))$. Then the $\lambda(\mathbf{p}(t))$ eigenvalues of the closed-loop varying parameter dependent system matrix and the $\lambda(\mathbf{p}_{ref})$ eigenvalues of the closed-loop reference system matrix become equal due to the matrix similarity [7]. Thus, $\forall \mathbf{p}(t)$ $\lambda(\mathbf{p}(t)) = \lambda(\mathbf{p}_{ref})$ if $\lambda(\mathbf{p}(t))$ denotes the eigenvalues of $(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}\mathbf{e}(t)))$. This can occur only if the similarity transformation matrix is the unity matrix with the same dimension, i.e. $\mathbf{I}_{n\times n}$. In this way the equality $\mathbf{A}_{ref} - \mathbf{B}\mathbf{K}_{ref} = \mathbf{I}^{-1}(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(t)e(t)))\mathbf{I}$ has to hold and the introduced complementary gain provide not just the smoother similarity, but also the strict equality criteria as well. Thus, the proposed complementary feedback gain $\mathbf{K}_{ref} + \mathbf{K}(t)e(t)$ provide the equality of the eigenvalues, i. e. $\lambda(\mathbf{p}_{ref}) = \lambda(\mathbf{p}(t))$.

Based on the equality of the eigenvalues of the closed-loop systems the $\mathbf{K}(t)$ matrix can be calculated for every occurring LTI system based on the actual values of the $\mathbf{p}(t)$:

$$\begin{aligned}\mathbf{K}(t) &= \frac{\mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{B}\mathbf{K}_{ref} - \mathbf{A}(\mathbf{p}(t)) + \mathbf{B}\mathbf{K}_{ref})}{-e(t)} \\ &= \frac{\mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))}{-e(t)}. \qquad (5)\end{aligned}$$

Thus, the control law becomes

$$\mathbf{u}(t) = (\mathbf{K}_{ref} + \mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))))\mathbf{x}(t). \quad (6)$$

This control law ensures that the dynamics of the closed-loop system will obey the dynamics prescribed for the reference system, i.e. the closed-loop system poles will coincide with the poles of $A_{ref} - BK_{ref}$. This implies that for any value of the parameter $p$, the closed-loop system will behave as a linear system whose dynamics is identical to the prescribed dynamics.

This methodology can be used to design a complementary observer structure as well. A full order linear observer can be described by the dynamics governed by the differential equation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + \mathbf{G}\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) \ , \quad (7)$$

where the observer state matrix is $\mathbf{F} \in \mathbb{R}^{n\times n}$, the observer gain matrix is $\mathbf{G} \in \mathbb{R}^{k\times n}$ and the observer input matrix is $\mathbf{H} \in \mathbb{R}^{n\times m}$. The observation error disappears with given velocity determined by the prescribed eigenvalues of $\mathbf{F}$ [10]:

$$|s\mathbf{I} - \mathbf{F}| = |s\mathbf{I} - \mathbf{A} - \mathbf{G}\mathbf{C}| = |s\mathbf{I} - \mathbf{A}^\top - \mathbf{C}^\top\mathbf{G}^\top| \ . \quad (8)$$

Consider that $\mathbf{F} = \mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} \sim \mathbf{F}(t) = \mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(t)e(t))\mathbf{C}$. That means that the eigenvalues of $\lambda(\mathbf{F}(t))$ and $\lambda(\mathbf{F})$ become equal during operation. Thus, $\forall\mathbf{p}(t)$ $\lambda(\mathbf{F}(t)) = \lambda(\mathbf{F})$ if $\lambda(\mathbf{F}(t))$ denotes the eigenvalues of $\mathbf{F}(t) = \mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(t)e(t))\mathbf{C}$. Similarly to the previous case, this is only possible if the similarity transformation matrix is the unity matrix $\mathbf{I}_{n\times n}$, i.e. $\mathbf{F} = \mathbf{I}^{-1}\mathbf{F}(t)\mathbf{I}$. As in the case of the complementary controller, this means that the introduced observer gain provides the similarity, and also the equality criteria as well. Equality of the eigenvalues can be rearranged to calculate $\mathbf{G}(t)$:

$$\begin{aligned}
&\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} = \mathbf{A}(\mathbf{p}(t)) - \mathbf{G}_{ref}\mathbf{C} - \mathbf{G}(t)\mathbf{C}e(t) \\
&(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} - \mathbf{A}(\mathbf{p}(t)) + \mathbf{G}_{ref}\mathbf{C})\mathbf{C}^{-1} \\
&\qquad\qquad = -\mathbf{G}(t)\mathbf{C}\mathbf{C}^{-1}e(t) \\
&\mathbf{G}(t) = \frac{(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} - \mathbf{A}(\mathbf{p}(t)) + \mathbf{G}_{ref}\mathbf{C})\mathbf{C}^{-1}}{-e(t)} \ . \quad (9) \\
&\mathbf{G}(t) = \frac{(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1}}{-e(t)}
\end{aligned}$$

Thus, the dynamics of the observer is governed by the differential equation

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + (\mathbf{G}_{ref} + (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) \quad (10)$$

With the proposed complementary LPV controller and observer structure it is possible to control different nonlinear system via LPV framework – without linearization, or using LMI-based methods.

In the current form of the design methodologies, the main limitation of the methods is that the system has to have the same number of control inputs as state variables and an invertible input matrix in order to be able to apply the control law (6), while the system has to have the same number of observable outputs as the number of state variables and an invertible output matrix in order to be able to apply the observer (10). Further details, limitations, consequences and examples regarding this controller and observer design method can be found in the cited literature [6], [8], [9].

### *C. Particular steps to realize complementary LPV controller and observer in practice*

Here we have collected the main steps of the realization of the complementary LPV controller and observer structure.

- Realization of the appropriate $\mathbf{S}(\mathbf{p}(t))$ LPV model form from the original nonlinear model;
- Selection of the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system (which is an underlying LTI system as well);
- Design of the $\mathbf{K}_{ref}$ reference state feedback controller with an arbitrary method which is appropriate to control the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system;
- Design of the $\mathbf{G}_{ref}$ reference observer gain with an arbitrary method which is appropriate to observe the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system;
- Realization of the complementary LPV controller (6) and observer structure (10).

## III. Controller design

### *A. An example nonlinear compartmental model*

Consider the following nonlinear compartmental model which is described by the differential equations

$$\begin{aligned}
\dot{x}_1(t) &= cx_1(t)x_2(t) - \frac{dx_1(t)}{1+ax_1(t)} + k_2x_2(t) + \frac{u_1(t)}{V_1} \\
\dot{x}_2(t) &= k_1x_3(t) - k_2x_2(t) + \frac{u_2(t)}{V_2} \\
\dot{x}_3(t) &= -\frac{k_3x_3(t)}{1+bx_3(t)} + k_1x_3(t) + \frac{u_3(t)}{V_3} \\
y_1(t) &= o_1x_1(t) \\
y_2(t) &= o_2x_2(t) \\
y_3(t) &= o_3x_3(t)
\end{aligned} \quad (11)$$

with $a = 0.05$ [L/mmol], $b = 0.35$ [L/mmol], $c = 0.4$ [1/min], $d = 0.2$ [1/min], $k_1 = 0.02$ [1/min], $k_2 = 0.1$ [1/min], $k_3 = 0.085$ [1/min], $V_1 = 1.5$ [L], $V_2 = 0.75$ [L], $V_3 = 2$ [L], $o_1 = 2$ [-], $o_2 = 0.5$ [-] $o_3 = 3$ [-]. The state variables are $x_1(t)$, $x_2(t)$ and $x_3(t)$ while the outputs are formed from the states after multiplication by scalars. The $u_1(t)$, $u_2(t)$ and $u_3(t)$ [mmol/min] are the inputs. The model contains three nonlinearities: the product of the $x_1(t)$ and $x_2(t)$ states and the natural degradations of the $x_1(t)$ and $x_3(t)$ compartments loaded by Michaelis-Menten type saturation. We can select the mentioned terms causing nonlinearity as scheduling variables: $p = \left[x_1(t), \frac{1}{1+ax_1(t)}, \frac{k_3}{1+bx_3(t)}\right]^\top$, which generates a $3D$ parameter space.

The state-space representation and the state matrices of the LPV system in affine LPV case can be written as follows:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \\ y_1(t) \\ y_2(t) \\ y_3(t) \end{pmatrix} = \mathbf{S}(\mathbf{p}(t)) \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{p}(t) & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ u_1(t) \\ u_2(t) \\ u_3(t) \end{pmatrix} \tag{12}$$

$$\mathbf{A}(\mathbf{p}(t) = \begin{bmatrix} 0 & k_2 & 0 \\ 0 & -k_2 & k_1 \\ 0 & 0 & -k_1 \end{bmatrix} + \begin{bmatrix} 0 & c & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} p_1(t) + \begin{bmatrix} 0 & 0 & 0 \\ -d & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} p_2(t) + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} p_3(t) \tag{13}$$

$$\mathbf{B} = \begin{bmatrix} 1/V_1 & 0 & 0 \\ 0 & 1/V_2 & 0 \\ 0 & 0 & 1/V_3 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Note that the $\mathbf{B}$ and $\mathbf{C}$ matrices are invertible, so the controller and observer design methodologies can be applied for this model.

### B. Complementary LPV Controller Design

Assume that $\mathbf{p}_{ref} = [0.5, 0.9756, 0.0794]^\top$ - in this case $x_{1,d} = 0.5$ and $x_{3,d} = 0.2$ - which is considered as an appropriate reference parameter vector with $\mathbf{S}(\mathbf{p}_{ref})$ being the reference LTI system. The state matrix is the only affected term and at the reference point $\mathbf{A}(\mathbf{p}_{ref})$ becomes

$$\mathbf{A}(\mathbf{p}_{ref}) = \begin{bmatrix} -0.1951 & 0.27 & 0 \\ 0 & -0.07 & 0.002 \\ 0 & 0 & 0.0774 \end{bmatrix}. \tag{14}$$

The eigenvalues of the reference state matrix $\mathbf{A}(\mathbf{p}_{ref})$ are $\lambda = [-0.1951, -0.07, 0.0774]^\top$, which means that the reference LTI system is unstable.

The $\mathbf{Co}$ controllability matrix shows that the system is controllable: $rank(\mathbf{Co}) = 3 \equiv n$, i.e. the reference LTI system is controllable. Thus, the reference state feedback-based controller design ( realization of the state-feedback $\mathbf{u} = -\mathbf{K}_{ref}\mathbf{x}$ ) is possible. Note that if the input matrix $\mathbf{B}$ is invertible, then the controllability is trivial, since $\mathbf{Co} = \mathbf{B}$.

For the design of optimal $\mathbf{K}_{ref}$ gain (LQ optimal control) the MATLAB© *care* command was used. We chose the design parameters $\mathbf{Q}$ and $\mathbf{R}$ as $\mathbf{Q} = \mathbf{C}\mathbf{C}^\top$ and $\mathbf{R} = \text{diag}(1/100, 1/100, 1/100)$.

The embedded *care* command calculates $\mathbf{X}$ as the unique solution of the continuous-time control algebraic Ricatti equation [11]:

$$\begin{aligned} &\mathbf{A}^\top\mathbf{X}\mathbf{E} + \mathbf{E}^\top\mathbf{X}\mathbf{A} \\ &\quad -(\mathbf{E}^\top\mathbf{X}\mathbf{B} + \mathbf{S})\mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{X}\mathbf{E} + \mathbf{S}^\top) + \mathbf{Q} = \mathbf{O} \end{aligned}, \tag{15}$$

where the calculated optimal gain – with $\mathbf{S} = \mathbf{0}$ and $\mathbf{E} = \mathbf{I}$ – is equal to $\mathbf{K}_{ref} = \mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{X}\mathbf{E} + \mathbf{S}^\top)$.

The calculated optimal gain is

$$\mathbf{K}_{ref} = \begin{bmatrix} 19.7024 & 0.2658 & 0 \\ 0.5315 & 4.9622 & 0.0005 \\ 0 & 0.0002 & 30.1553 \end{bmatrix}. \tag{16}$$

The eigenvalues of the closed-loop reference system matrix $\mathbf{A}(\mathbf{p}_{ref}) - \mathbf{B}\mathbf{K}_{ref}$ are $\lambda_{ref,closed} = [-6.6962, -13.3201, -15.0002]^\top$ – i.e. the negative real eigenvalues provide stability and fast transient.

The structure of the complementary controller provides the aforementioned strict equality, namely $\lambda_{LPV,closed} = \lambda_{ref,closed}$ everywhere regardless the actual value of the $\mathbf{p}(t)$ parameter vector. At this point, the complementary LPV controller can be realized using (6).

### C. Complementary LPV Observer Design

The complementary LPV observer can be calculated similarly as the controller. The observability matrix shows that realization of state observer for the reference LTI system is possible, since $rank(\mathbf{Ob}) = 3 \equiv n$, i.e. the system is fully observable. Note that since the output matrix $\mathbf{C}$ is invertible, the observability is trivial, since $\mathbf{Ob} = \mathbf{C}$.

The reference observer gain $\mathbf{G}_{ref}$ was designed via MATLAB© *place* command [11] with describing the desired eigenvalues of the system matrix characterizing the observer error dynamics to be $\lambda_{obs} = [-40, -40, -40]^\top$ (that results in faster dynamics than the dynamics of the closed-loop system $\lambda_{ref,closed}$). The resulting $\mathbf{G}_{ref}$ is

$$\mathbf{G}_{ref} = 10^3 \begin{bmatrix} 0.1348 & -0.0072 & -0 \\ -0.0145 & 1.0745 & -0 \\ 0 & -0 & 0.0665 \end{bmatrix}. \tag{17}$$

We need to use additional feed forward compensators as well in order to reach the desired steady-state of the output. These feed forward compensator matrices should be $\mathbf{p}(t)$-dependent as well. The calculation of feed forward compensator matrices can be found in [8], [9]. One compensator modifies the state vector by substracting the desired steady-state from the actual state of the system, the steady-state is calculated as $N_x(\mathbf{p}(t))r(t)$, where $r(t)$ is the reference signal in the time instant $t$, and the other modifies the control input by adding the steady-state control input calculated as $N_u(\mathbf{p}(t))r(t)$. Thus, the controller is governed by the equations

$$\begin{aligned} \dot{\hat{\mathbf{x}}}(t) &= \mathbf{F}\hat{\mathbf{x}}(t) + (\mathbf{G}_{ref} + (A_{ref} - A(\mathbf{p}(t)))C^{-1})\mathbf{y}(t) \\ &\quad +\mathbf{H}\mathbf{u}(t) \end{aligned} \tag{18}$$

$$\begin{aligned} \mathbf{u}(t) &= (K_{ref} + B^{-1}(A_{ref} - A(\mathbf{p}(t)))) \cdot \\ &\quad \cdot(\hat{\mathbf{x}}(t) - N_x(\mathbf{p}(t))r(t)) + N_u(\mathbf{p}(t))r(t). \end{aligned} \tag{19}$$

## IV. Results

In this section we carry out simulations and apply the designed controller and observer for different system models (reference system, LPV model and the original nonlinear model), and compare the results. The simulations are carried out with the following system models:

1) The reference LTI system $\mathbf{S}_{ref}$: state vector $\mathbf{x}_r(t)$, output vector $\mathbf{y}_r(t)$, the permanent parameter vector $\mathbf{p}_{ref}$ and the control signal $\mathbf{u}_r(t)$ generated by the state feedback reference controller $\mathbf{K}_{ref}$ and constant feed forward compensators;
2) The LPV system $\mathbf{S}(\mathbf{p}(t))$ without observer (supposing that the states can be measured): state vector $\mathbf{x}_{lpv}(t)$, output vector $\mathbf{y}_{lpv}(t)$, parameter vector $\mathbf{p}_{lpv}(t)$ and the control signal $\mathbf{u}_{lpv}(t)$ generated by the complementary LPV controller with control law (19) with $\hat{\mathbf{x}}$ replaced by $\mathbf{x}_{lpv}$;
3) LPV system $\mathbf{S}(\mathbf{p}(t))$ with complementary LPV observer: observed state vector $\hat{\mathbf{x}}_{lpvobs}(t)$ coming from the complementary LPV observer (10), output vector of the LPV system $\mathbf{y}_{lpvobs}(t)$, parameter vector $\mathbf{p}_{lpvobs}(t)$ generated by the observed states $\hat{\mathbf{x}}_{lpvobs}(t)$ and the control signal $\mathbf{u}_{lpvobs}(t)$ generated by the complementary LPV controller with control law (19);
4) Original nonlinear system with complementary LPV observer: observed state vector $\hat{\mathbf{x}}_{nonobs}(t)$ coming from the complementary LPV observer (10), output vector of the LPV system $\mathbf{y}_{nonobs}(t)$, parameter vector $\mathbf{p}_{nonobs}(t)$ generated by the observed states $\hat{\mathbf{x}}_{nonobs}(t)$ and the control signal $\mathbf{u}_{nonobs}(t)$ generated by the complementary LPV controller with control law (19).

During the simulation we used the same settings in every cases. The reference signal for the system outputs is $\mathbf{r} = [16, 2.5, 30]^\top$, thus we want the steady-state of the system outputs to be $\mathbf{y}_\infty = \mathbf{r}$. The corresponding steady-state system state vector is $\mathbf{x} = [8, 5, 10]^\top$. The initial state vector for every system was $\mathbf{x}(0) = [3, 2, 1]^\top$. The initial state vector for the observers was $\hat{\mathbf{x}}(0) = [4, 4, 4]^\top$ – in this way there is an initial observation error, thus the dynamics of the observer can be analyzed. The simulation time was 1 min – all the transients disappear after 1 minute because of the fast system dynamics.

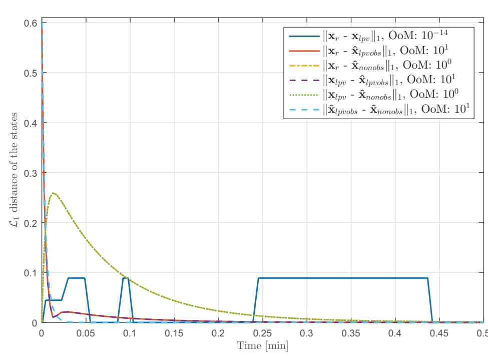


Figure 1: Comparison of the states during simulation. [OoM – Order of Magnitude]

The signals of the different systems are compared on Figs. 1-4. The absolute values of the differences of different signals are depicted in order to compare the different system transients. Since most of the signals are not scalars, but vectors, we consider the sum of the absolute values of the components,

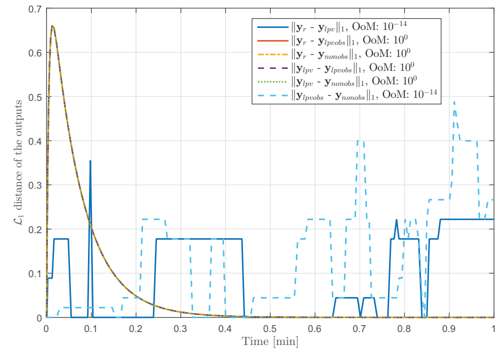


Figure 2: Comparison of the outputs during simulation. [OoM – Order of Magnitude]

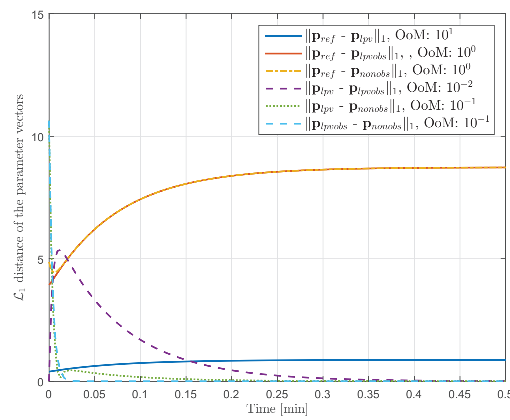


Figure 3: Comparison of the parameter vectors during simulation. [OoM – Order of Magnitude]

i.e. the $\mathcal{L}_1$ norm of the vectors in each time instant. Due to the lack of space in the paper, we use data standardization to be able to depict signals having amplitude of different order of magnitude in the same figure. We have given the order of magnitude (OoM) for each curve in the corresponding legend.

Figure 1. shows the comparison of the states. The difference of the states of the reference LTI system and the states of the LPV system is very small (OoM of $10^{-14}$), they are practically equal, so the LPV system works the same way as the reference LTI system, as desired. Since the observer is initiated from a different state than the original initial state of the system, the observed states differ from the real states initially, however this difference tends to zero thus the observers are stable. The same results for the outputs of the system can be examined in Fig. 2.

Figure 3. shows the difference of the parameter vectors for the different systems. By examining the alteration from the constant $p_{ref}$ vector we can see how far the other systems are from the reference LTI system. The difference of the parameters of the systems with and without observer tend to zero, since the observers are stable.

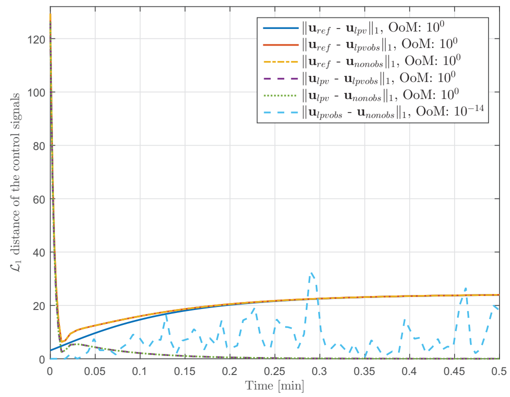


Figure 4: Comparison of the control signals during simulation. [OoM – Order of Magnitude]

Similar behavior can be examined for the control signals in Fig. 4. The difference of the control signals for the reference LTI system and the other systems are large, since the other systems alter from the reference LTI systems as well. However, the control signals for the LPV and nonlinear systems are very similar. Note that the control signal for the LPV system with observer and the original nonlinear system with observer is almost the same, the difference has OoM of $10^{-14}$.

Figure 5. shows the trajectories of the LTI system with and without observer. The start points of these trajectories are different, since the observer has an estimation error in the initial time, however the trajectories unite after a short time since the observers are stable. The lower part of Fig. 5 shows the same for the observer and real parameter values, demonstrating that the LPV control methodology can be applied for the given system, since the parameters required for the LPV model can be acquired using the state estimator. Note that each value of the parameter vectors represents an LTI system, this is why we labeled the initial and end points with the notation $\mathbf{S}(\mathbf{p})$ in the legends.

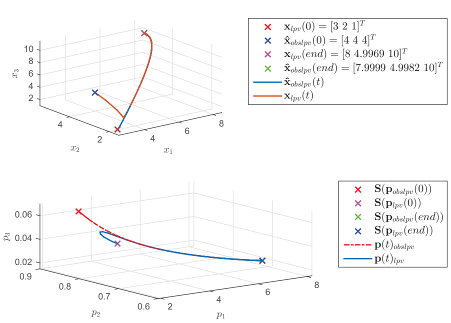

Figure 5: Results of the simulation in case of the LPV system with and withour observer [Upper diagram: Phase space of the Observed states and LPV system's states; Lower diagram: $3D$ parameter space during simulations with varying parameter vectors provided by the observer and the LPV system itself]

## V. Conclusion

The methodology presented here can be used to control and observe a nonlinear system that can be modeled with LPV technique, and it can be guaranteed that the resulting closed-loop system dynamics is the same as a reference LTI system. The simulation results show that the technique works well for a nonlinear system that can be modeled as an LPV system. The main drawbacks of the methodology are that it requires invertible input and output matrices; generalizing the results for noninvertible input and output matrices is subject to further research.

The invertibility of the output matrix also means that the states of the system can be calculated algebraically from the observed outputs, i.e. $x = C^{-1}y$, however this process is sensitive to measurement noises. The observer presented here can be used to filter out these noises by the integration and considering the information about the dynamics of the system.

The presented approach is similar to exact linearization [12], [13] in nature since it linearizes the originally nonlinear system using state-feedback. However, exact linearization transforms the closed-loop system into a series of integrators, the presented methodology transforms the system to a linear system with arbitrary dynamics.

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