Control of Physiological Systems through Linear Parameter Varying Framework

György Eigner

Physiological Controls Research Center
Research and Innovation Center of Óbuda University
H-1032, Budapest, Kiscelli street 82
eigner.gyorgy@nik.uni-obuda.hu

Abstract: The paper presents a novel controller and observer design methodology for nonlinear systems based on the Linear Parameter Varying (LPV) framework. The introduced techniques effectively combine the classical state feedback methodology with matrix similarity theorems. The presented solutions are analyzed in order to assess their benefits, drawbacks and limitations. The possible selection of scheduling variables is investigated and dyadic structures are used to strengthen the eigenvalue equality from a mathematical point of view. The connection between the controller and observer side is presented and a solution is given for occurring matrix invertibility issues. The method is tested for a control of nonlinear physiological system, more specifically, for the control of innate immune system. The results show that the developed complementary LPV controller and observer are able to satisfy the predefined criteria.

Keywords: Linear Parameter Varying, Control of immune system, LPV-based control technique, Physiological control

1 Introduction

During controller design, today's control engineers have to face many challenges. On the one hand, application of nonlinear controller design techniques requires an experienced designer, use of advanced mathematical tools and unique approaches in each case. On the other hand, the application of well-known linear design methods provides controllers operating only under specific circumstances. Since the real world processes are not linear, to catch that specific proper operating environment in which the nonlinear process can be handled as linear is difficult [1].

In the last two decades, several applications appeared – mostly on LPV basis – that aim to deal with the effective combination of the linear controller design methods on nonlinear systems under given restrictions and requirements. Besides, such innovative methods appeared that effectively exploit the iterative and adaptive techniques, even without the use of the Lyapunov laws or other techniques.

A good example for the latter is the Robust Fixed Point Transformation (RFPT) based techniques. The RFPT-based observer design formalizes the control problem as a fixed point problem. By using adaptive iterative mathematical tools the solution of the fixed point problem also becomes the desired control action which satisfies the predefined criteria [2, 3].

An important direction in the combination of linear design methods on a Lyapunov basis and nonlinear control tasks is the application of the LPV framework [4]. All state space models can be described as LPV models in which the most crucial properties are represented by the so-called parameter vector. A LPV model consists of finite or infinite Linear Time Invariant (LTI) systems. That is, which LTI system's properties reflect in the LPV system depends on the varying parameter vector. If the parameter vector is constant then the LPV system reduces to a LTI system.

Many application possibilities appeared recently which effectively exploited the benefits of the LPV model descriptions [5, 6]. One of these is the Gain Scheduling (GS) controller design [7, 8]. In this case the parameter space of the LPV system is divided into sections and it is possible to design such controllers on the basis of linear design techniques that can deal with the control of a given sector. In this way there are plenty different, but similar controllers designed one-by-one for each sector. The change between the designed sector controllers is taking place accordingly the variation of the parameter vector. Another direction is the controller design for polytopic LPV models via Linear Matrix Inequalities (LMI). In this case, the resulting controller can be designed for a given parameter domain – defined as a hyperbox in the parameter space – by using LMI techniques. The control tasks can be formalized as a LMI, which satisfies given prescriptions and true for all vertices of the defined polytope. Via optimization it is possible to design such a LMI-LPV controller, which is the convex combination of the designed subcontrollers (one for each vertex) and, which is able to control the system, if the parameter vector is inside the given domain [9, 10]. There are other possibilities as well which aim to catch all possible occurring LTI systems during the operation, like the frequency domain based methods [11]. Although, all of them have many benefits, but they do have drawbacks as well. One main limitation is that these methods use only a particular region of the parameter space and do not provide a solution for the whole.

In this work we focus on another direction, namely, we aim to provide a controller design solution that is able to handle the whole parameter space beside the appropriate control action and global stability. The introduced method uses the mathematical properties of the parameter space of the LPV systems and linear controller design techniques. Furthermore, it does not require LMIs or other computational costly methods.

The paper is organized as follows: first we introduce the LPV based design method, mathematical tools, limitations, applicability and developed control structure. Afterwards, the application of the method in case a physiological system is shown. Finally, we conclude with the results and give a short outline of the future work.

2 The LPV-based Design Method

2.1 LPV Systems in General

In this section the developed state feedback based complementary LPV controller and observer designs are detailed. The procedure allows the controller design for the nonlinear system to be controlled – given by its state-space representation – through the so-called LPV framework. The method combines modern state feedback design, LPV methods and the matrix similarity theorems in order to realize the complementary LPV controller and observer.

Definition 1. *LPV model in state space form*
A LPV model can be described in state space representation, and the compact form of it is:

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t)+\mathbf{B}(\mathbf{p}(t))\mathbf{u}(t)+\mathbf{E}(\mathbf{p}(t))\mathbf{d}(t)\\ \mathbf{y}(t) &= \mathbf{C}(\mathbf{p}(t))\mathbf{x}(t)+\mathbf{D}(\mathbf{p}(t))\mathbf{u}(t)+\mathbf{D_2}(\mathbf{p}(t))\mathbf{n}(t)\end{aligned} \tag{1a}$$

$$\mathbf{S}(\mathbf{p}(t)) = \begin{pmatrix}\mathbf{A}(\mathbf{p}(t)) & \mathbf{B}(\mathbf{p}(t)) & \mathbf{E}(\mathbf{p}(t)\\ \mathbf{C}(\mathbf{p}(t)) & \mathbf{D}(\mathbf{p}(t)) & \mathbf{D_2}(\mathbf{p}(t))\end{pmatrix}\ , \tag{1b}$$

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \mathbf{S}(\mathbf{p}(t))\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\\ \mathbf{d}(t)\\ \mathbf{n}(t)\end{pmatrix}\ , \tag{1c}$$

where $\mathbf{A}(\mathbf{p}(t)) \in \mathbb{R}^{n\times n}$ is the state matrix, $\mathbf{B}(\mathbf{p}(t)) \in \mathbb{R}^{n\times m}$ is the control input matrix, $\mathbf{E}(\mathbf{p}(t)) \in \mathbb{R}^{n\times h}$ is the disturbance input matrix, $\mathbf{C}(\mathbf{p}(t)) \in \mathbb{R}^{k\times n}$ is the output matrix, $\mathbf{D}(\mathbf{p}(t)) \in \mathbb{R}^{k\times m}$ is the control input forward matrix and $\mathbf{D_2}(\mathbf{p}(t)) \in \mathbb{R}^{k\times h}$ disturbance input forward matrix. Moreover, $\mathbf{u}(t) \in \mathbb{R}^m$, $\mathbf{d}(t) \in \mathbb{R}^h$, $\mathbf{n}(t) \in \mathbb{R}^h$, $\mathbf{y}(t) \in \mathbb{R}^k$ and $\mathbf{x}(t) \in \mathbb{R}^n$ vectors are the control, disturbance and noise inputs, output and state vector, respectively.
$\mathbf{S}(\mathbf{p}(t)) \in \mathbb{R}^{(n+k)\times(n+m+h)}$ is the parameter dependent system matrix, which equivocally determines the LPV system. Further, the $\mathbf{p}(t) \in \Omega \in \mathbb{R}^q$ is the time dependent parameter vector.

Evidently, if a LPV system does not contain or model the noise and disturbance then only $\mathbf{A}(\mathbf{p}(t))$, $\mathbf{B}(\mathbf{p}(t))$, $\mathbf{C}(\mathbf{p}(t))$ and $\mathbf{D}(\mathbf{p}(t))$ matrices occur and $\mathbf{S}(\mathbf{p}(t))$ consists of these matrices in appropriate dimensions.

Definition 2. *Parameter vector and parameter space*
The $\mathbf{p}(t) \in \Omega \in \mathbb{R}^q$ real parameter vector consists of the so-called scheduling variables $p_i(t)$ $i = 1,2,\ldots,q$, which are selected terms of the original nonlinear model. The $\mathbf{p}(t)$ spans the $\mathbb{R}^q$ real parameter space (which is a real Euclidean vector space) in which the dimension q is equal to the number of the selected scheduling variables (dimension of the parameter vector). The $\Omega \in \mathbb{R}^q$ is a bounded subspace (hypercube) of the parameter space that is determined by the interpreted (reasonable/possible) extremes of the scheduling variables, i.e. $\mathbf{p}(t)$: $\Omega = [p_{1,min}, p_{1,max}] \times$

$[p_{2,min}, p_{2,max}] \times \ldots \times [p_{q,min}, p_{q,max}] \in \mathbb{R}^q$. *The variation of the* $\mathbf{p}(t)$ *must be inside* Ω, *if* Ω *is defined.*

Remark 1. qLPV model
If the $\mathbf{p}(t)$ does contain not only scalers or functions from the original nonlinear model but state variables as well it is called quasi-LPV (qLPV) model.

Usually, the LPV system can be generally described only in affine and polytopic forms [9]. In this study, the general LPV form is used, which means the $\mathbf{p}(t)$ is embedded directly into the system matrices.

Remark 2. Selection of the $p_i(t)$ $i = 1, 2, \ldots, q$ scheduling variables
The $p_i(t)$ scheduling variables should be the nonlinearity inducing terms in the original nonlinear system. In this way, the $\mathbf{p}(t)$ contains each nonlinearity inducing elements from the system equation, thus the $\mathbf{S}(\mathbf{p}(t))$ LPV description is able to hide the nonlinear terms and handle them as scalars (if $\mathbf{p}$ is fixed) or time varying parameters (if $\mathbf{p}(t)$ varies in time). Appropriate selection of $p_i(t)$ is a key condition for controllability and observability of the later defined reference LTI system.

2.2 State Feedback, Controllability and Observability

The applicability of a state feedback based controller depends on the controllability (stabilizability) and observability (detectability) of the given system. These criteria – due to Kalman [12] – are determined by the structures of the given system representation. More precisely, the eigenvalues of $\mathbf{A}$ (modes of the system), the $\mathbf{B}$ input matrix and the $\mathbf{C}$ output matrix determine these key properties, if disturbance, noise and direct coupling between the input and output are not considered [13, 14].
Consider the following dynamical LTI system:

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \quad . \tag{2}$$

The system (2) is controllable, if the $\mathbf{Co} = [\mathbf{B} \;\; \mathbf{AB} \;\; \mathbf{A}^2\mathbf{B} \;\; \ldots \;\; \mathbf{A}^{n-1}\mathbf{B}]$ controllability matrix has full row rank, or equivalently, if all modes of $\mathbf{A}$ ($\lambda(\mathbf{A})$ eigenvalues) are accessible through $\mathbf{B}$, namely $v^*\mathbf{A} = \lambda(\mathbf{A})v^*$ and $v^*\mathbf{B} \neq 0$ (the latter criteria is the so-called Popov-Belevitch-Hautus (PBH) test) [14]. In this case, it is possible to design such $\mathbf{K}$ feedback gain through which the $\mathbf{A} - \mathbf{BK}$ closed-loop poles $\lambda(\mathbf{A} - \mathbf{BK})$ can be freely assigned on the complex plain and the unstable modes can be stabilized, i.e. $\mathbf{u}(t) = -\mathbf{K}\mathbf{x}(t)$ and

$$\begin{aligned} \dot{\mathbf{x}}(t) &= (\mathbf{A} - \mathbf{BK})\mathbf{x}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \quad . \tag{3}$$

The system (2) is observable, if the $\mathbf{Ob} = [\mathbf{C} \;\; \mathbf{CA} \;\; \mathbf{CA}^2 \;\; \ldots \;\; \mathbf{CA}^{n-1}]^\top$ observability matrix has full column rank, or equivalently, if all modes of $\mathbf{A}$ ($\lambda(\mathbf{A})$ eigenvalues) are detectable through $\mathbf{C}$, namely $\mathbf{A}w = \lambda(\mathbf{A})w$ and $\mathbf{C}w \neq 0$ [14].
In this case, it is possible to design such $\mathbf{G}$ observer gain through which the $\mathbf{A} -$

$\mathbf{GC}$ closed-loop poles $\lambda(\mathbf{A}-\mathbf{GC})$ can be freely assigned on the complex plain and $e(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ observation error $e(t) \to 0$, $t \to \infty$.

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A}-\mathbf{GC})\hat{\mathbf{x}}(t) + \mathbf{Gy}(t) + \mathbf{Hu}(t) \ , \tag{4}$$

where $\mathbf{H} := \mathbf{B}$.
Assume that the $\mathbf{p}(t)$ parameter vector is fixed (does not vary in time) and named as $\mathbf{p}_{ref}$ reference parameter vector. In this case, the $\mathbf{S}(\mathbf{p}(t))$ general LPV system simplifies to a $\mathbf{S}_{ref} := \mathbf{S}(\mathbf{p}_{ref})$ reference LTI system. Controllability and observability of the reference LTI system is a key property according to the preliminary assumptions.

Remark 3. Before we go further two limitations should be noted regarding this study:

- Only fully controllable and observable $\mathbf{S}_{ref}$ reference LTI systems are investigated – investigation of only stabilizable and detectable systems will be the part of the future work;
- Parameter dependency can occur only in the $\mathbf{A}(\mathbf{p}(t))$ system matrix – thus, other matrices cannot contain parameter dependent terms.

The latter restriction can be easily relaxed. If an input (output) contains nonlinearity causing element this input (output) should be handled as new "state variable" and with the extension of $\mathbf{A}$ and reduction of $\mathbf{B}$ ($\mathbf{C}$) the term can be linked to a state (e.g. input case: $\dot{x}_1(t) = x_1(t)u_1(t) \to \dot{x}_1(t) = x_1(t)x_2(t)$ and $\dot{x}_2(t) = u(t)$; output case: $y(t) = x_1(t)x_2(t) \to x_3(t) = x_1(t)x_2(t)$ and $y(t) = x_3(t)$). The price is the extra dynamics, which has to be handled.
The controllability depends on the $(\mathbf{A}(\mathbf{p}_{ref}), \mathbf{B})$ complex. Assume that the reference controllability matrix $\mathbf{Co}_{ref} = [\mathbf{B} \ \ \mathbf{A}(\mathbf{p}_{ref})\mathbf{B} \ \ \mathbf{A}(\mathbf{p}_{ref})^2\mathbf{B} \ \ \dots \ \ \mathbf{A}(\mathbf{p}_{ref})^{n-1}\mathbf{B}]$. If $\mathrm{rank}(\mathbf{Co}_{ref}) = n$ then the $(\mathbf{A}(\mathbf{p}_{ref}), \mathbf{B})$ (thus, the reference LTI system) is controllable.
The observability depends on the $(\mathbf{A}(\mathbf{p}_{ref}), \mathbf{C})$ complex. Assume that the reference observability matrix $\mathbf{Ob}_{ref} := [\mathbf{C} \ \ \mathbf{CA}(\mathbf{p}_{ref}) \ \ \mathbf{CA}(\mathbf{p}_{ref})^2 \ \ \dots \ \ \mathbf{CA}(\mathbf{p}_{ref})^{n-1}]^\top$. If $\mathrm{rank}(\mathbf{Ob}_{ref}) = n$ then the $(\mathbf{A}(\mathbf{p}_{ref}), \mathbf{B})$ (thus, the reference LTI system) is observable.

Remark 4. It is important to realize that $\mathbf{S}(\mathbf{p}(t))$ LPV system cannot be controlled and/or observed at every $\mathbf{p}(t)$ (everywhere in the parameter domain). This can occur when $\mathbf{p}(t)$ or given elements of it become equal to zero. In this case, the rank of $\mathbf{Co}$ and/or $\mathbf{Ob}$ can be lower than n. Moreover, $\mathbf{p}(t)$ or $p_i(t)$ can cause linear dependencies in $\mathbf{Co}$ and/or $\mathbf{Ob}$ which reduces the rank of the matrices as well and reduces the controllability and/or observability.

Appropriate selection of $p_i(t)$ scheduling variables and the $\mathbf{p}_{ref}$ is critical and determines the controllability and observability properties of $\mathbf{S}_{ref}$. On the one hand, only those states can be embedded into the $\mathbf{p}(t)$ which can be measured or estimated – since the $\mathbf{p}(t)$ is directly used in the complementary controller and observer structures. On the other hand, the "positions" of the $p_i(t)$ scheduling variables in $\mathbf{A}(\mathbf{p}(t))$ are also crucial as we see later and determines which states can be linked to the scheduling variables.

2.3 Matrix similarity theorems and dyadic structures

The core of the developed complementary LPV controller and observer structures are based on the special properties of the similarity theorems. Further, dyadic structures are useful for the generalization of the methods as well. The following definitions, theorems and proofs can be found in [15, 16, 17, 18].

Definition 3. *Similarity of matrices:*
A quadratic, $n \times n$ matrix $\mathbf{Q}$ is similar to a matrix $\mathbf{W}$, if exists an invertible $\mathbf{R}$ matrix that is $\mathbf{Q} = \mathbf{R}^{-1}\mathbf{W}\mathbf{R}$. Notation: $\mathbf{Q} \sim \mathbf{W}$.

Theorem 1. *Similarity invariance of the determinants of matrices: If $\mathbf{Q} \sim \mathbf{W}$, then $|\mathbf{Q}| = |\mathbf{W}|$.*

Proof. Let $\mathbf{Q} \sim \mathbf{W}$, namely, $\mathbf{Q} = \mathbf{R}^{-1}\mathbf{W}\mathbf{R}$. Then $|\mathbf{Q}| = |\mathbf{R}^{-1}\mathbf{W}\mathbf{R}| = |\mathbf{R}^{-1}||\mathbf{W}||\mathbf{R}| = |\mathbf{W}|$, since $|\mathbf{R}||\mathbf{R}^{-1}| = 1$. [15, 17]. ■

Theorem 2. *If $\mathbf{Q} \sim \mathbf{W}$, then the characteristic polynomials of the matrices and thus, the eigenvalues and the geometric and algebraic multiplicities of the eigenvalues of the matrices are the same (i.e. $\lambda(\mathbf{Q}) = \lambda(\mathbf{W})$)*

Proof. Let $\mathbf{Q} \sim \mathbf{W}$, namely, $\mathbf{Q} = \mathbf{R}^{-1}\mathbf{W}\mathbf{R}$. Then $\mathbf{Q} - \lambda\mathbf{I} = \mathbf{R}^{-1}\mathbf{W}\mathbf{R} - \lambda\mathbf{R}^{-1}\mathbf{I}\mathbf{R} = \mathbf{R}^{-1}(\mathbf{W}\mathbf{R} - \lambda\mathbf{I}\mathbf{R}) = \mathbf{R}^{-1}(\mathbf{W} - \lambda\mathbf{I})\mathbf{R}$, namely, $\mathbf{Q} - \lambda\mathbf{I} \sim \mathbf{W} - \lambda\mathbf{I}$, where $\mathbf{I}$ is the unity matrix in appropriate dimension [15, 16]. ■

Definition 4. *Dyadic product (or shortly dyad):*
The product of $\mathbf{q}_{n\times 1}$ and $\mathbf{w}^{\top}_{1\times m}$ vectors results a $\mathbf{q}_{n\times 1}\ \mathbf{w}^{\top}_{1\times m} := \mathbf{X}_{n\times m}$ matrix [18].

Definition 5. *Sum of dyadic series:*
The sum of a dyadic series can be described with a product two matrices and the opposite is also true.

$$\begin{bmatrix} \\ \mathbf{q}_1 \\ \\ \end{bmatrix} \begin{bmatrix} & \mathbf{w}_1^{\top} & \end{bmatrix} + \vdots + \begin{bmatrix} \\ \mathbf{q}_k \\ \\ \end{bmatrix} \begin{bmatrix} & \mathbf{w}_k^{\top} & \end{bmatrix} = \sum_{i=1}^{k} \mathbf{q}_i \mathbf{w}_i^{\top} = \mathbf{Q}\mathbf{W}^{\top}$$

$$\mathbf{Q}\mathbf{W}^{\top} = \left[\begin{array}{c:c:c:c} \mathbf{q}_1 & \mathbf{q}_2 & \dots & \mathbf{q}_k \end{array}\right] \left[\begin{array}{c} \mathbf{w}_1^{\top} \\ \hdashline \mathbf{w}_2^{\top} \\ \hdashline \vdots \\ \hdashline \mathbf{w}_k^{\top} \end{array}\right] \tag{5}$$

Definition 6. *Minimal dyadic decomposition:*
If we realize a matrix as the sum of the minimum of dyads as possible [18].

Definition 7. *Rank of a matrix*
The rank of a matrix is equal to the number of dyads which are represented in the minimal dyadic decomposition [18].

On one hand, these mathematical tools can be used to define eigenvalues equality rules for state feedback systems. Further, the useful properties of dyadic representation – especially the rank criteria – can be used for generalization purposes for the developed method.

2.4 Design of Complementary LPV Controller

Assume that a state feedback controller for the reference $\mathbf{S}_{ref}$ can be designed which requires the satisfaction of the controllability criteria detailed above. In this case, the control law can be described as $\mathbf{u}(t) = -\mathbf{K}_{ref}\mathbf{x}(t)$. As we see in (3) the closed-loop system matrix becomes $\mathbf{A}_{ref} - \mathbf{BK}_{ref}$ whose eigenvalues $\lambda(\mathbf{A}_{ref} - \mathbf{BK}_{ref})$ define the dynamics of the reference system. In that case, if we want to use the state feedback control concerning a given LPV system the parameter dependency has to be represented in the state feedback controller as well. Thus, only $\mathbf{A}(\mathbf{p}(t))$ can be parameter dependent – as it was declared above – a given LPV system under control can be described as:

$$\begin{array}{l}\dot{\mathbf{x}}(t) = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{BK}(t)\big)\mathbf{x}(t) \\ \mathbf{y}(t) = \mathbf{Cx}(t)\end{array} \quad . \tag{6}$$

Definition 8. *Complementary* $\mathbf{p}(t)$ *dependent feedback gain*
A LPV feedback gain $\mathbf{K}(t)$ *consists of a* $\mathbf{K}_{ref}$ *reference and* $\mathbf{K}(t)$ *varying feedback gain as follows:* $\mathbf{K}(t) := \mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t))$. *Therefore, from (6)* $\big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{BK}(t)\big) = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t)))\big)$.

Assume that $\mathbf{A}_{ref} - \mathbf{BK}_{ref} \sim \mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t)))\ \ \forall \mathbf{p}(t)$. The eigenvalues (poles) of the closed-loop reference LTI system are $\lambda_{ref} := \lambda(\mathbf{A}_{ref} - \mathbf{BK}_{ref})$ and the eigenvalues (poles) of the closed-loop LPV system are $\lambda(\mathbf{p}(t)) := \lambda(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t))))$.
Theorem 2 consequences that $\lambda_{ref} = \lambda(\mathbf{p}(t))\ \ \forall \mathbf{p}(t)\ t \geq 0$ due to the similarity. From control perspective, this means that the controlled reference LTI system and the controlled LPV system will have the same eigenvalues (poles) everywhere in the parameter domain – which entails that they will have the same dynamics and behavior.
When the similarity transformation matrix is the $\mathbf{I}$ unity matrix in appropriate dimension, then similarity described above occurs as $\mathbf{A}_{ref} - \mathbf{BK}_{ref} = \mathbf{I}^{-1}\big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t)))\big)\mathbf{I}$. This equality provides not just the similar dynamical behavior, but also the possibility to compute the parameter dependent $\mathbf{K}(\mathbf{p}(t))$ on the Ω parameter domain at every $\mathbf{p}(t)\ t \geq 0$.

$$\begin{array}{l}\mathbf{A}_{ref} - \mathbf{BK}_{ref} = \mathbf{I}^{-1}\big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t)))\big)\mathbf{I} = \\ = \mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t))) \\ \mathbf{K}(\mathbf{p}(t)) = -\mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{BK}_{ref} - \mathbf{A}(\mathbf{p}(t)) + \mathbf{BK}_{ref}) = -\mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\end{array} \quad . \tag{7}$$

Hence, the control law becomes:

$$\mathbf{u}(t) = -(\mathbf{K}_{ref} + \mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))))\mathbf{x}(t) \quad . \tag{8}$$

Therefore, the (6) should be modified accordingly to (7) which leads back to (3) as we can see below due to the equality criteria.

$$\begin{aligned}\dot{\mathbf{x}}(t) = \; & \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} + \mathbf{K}(\mathbf{p}(t))\big)\mathbf{x}(t) = \\ & = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}(\mathbf{K}_{ref} - \mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\big)\mathbf{x}(t) = \\ & = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}\mathbf{K}_{ref} + \mathbf{B}\mathbf{B}^{-1}\mathbf{A}_{ref} - \mathbf{B}\mathbf{B}^{-1}\mathbf{A}(\mathbf{p}(t)))\big)\mathbf{x}(t) = \\ & = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}\mathbf{K}_{ref} + \mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))\big)\mathbf{x}(t) = \\ & = \big(\mathbf{A}_{ref} - \mathbf{B}\mathbf{K}_{ref}\big)\mathbf{x}(t) \\ \mathbf{y}(t) = \; & \mathbf{C}\mathbf{x}(t)\end{aligned} \quad . \tag{9}$$

2.5 Design of Complementary LPV Observer

Assume that a state observer for the reference $\mathbf{S}_{ref}$ can be designed which requires the satisfaction of the observability criteria detailed above. According to (4) the closed-loop system matrix becomes $\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C}$. The eigenvalues $\lambda(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C})$ define the dynamics of the reference observer. Similar to the control perspective, if we want to use the state observer regarding a given LPV system the parameter dependency has to be represented in the state observer as well. As previously, it is considered that only $\mathbf{A}(\mathbf{p}(t))$ can be parameter dependent and a given observed LPV system can be described as:

$$\dot{\hat{\mathbf{x}}}(t) = \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{G}(t)\mathbf{C}\big)\hat{\mathbf{x}}(t) + \mathbf{G}(t)\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) \quad . \tag{10}$$

Definition 9. *Complementary* $\mathbf{p}(t)$ *dependent observer gain*
A LPV observer gain $\mathbf{G}(t)$ *consists of a* $\mathbf{G}_{ref}$ *reference and* $\mathbf{G}(\mathbf{p}(t))$ *varying observer gain as follows:* $\mathbf{G}(t) := \mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t))$. *Therefore, from (10)* $\big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{G}(t)\mathbf{C}\big) = \big(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t))\mathbf{C}\big)$.

Assume that $\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} \sim \mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t)))\mathbf{C} \;\; \forall \mathbf{p}(t)$. The eigenvalues (poles) of the closed-loop reference LTI system are $\lambda_{ref} := \lambda(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C})$ and the eigenvalues (poles) of the closed-loop LPV system are $\lambda(\mathbf{p}(t)) := \lambda(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t)))\mathbf{C})$.
According to Theorem 2 the consequence of the similarity the $\lambda_{ref} = \lambda(\mathbf{p}(t)) \;\; \forall \mathbf{p}(t)$ $t \geq 0$. From control perspective point of view that means the reference LTI and the LPV observers will have the same eigenvalues (poles) everywhere in the parameter domain – which entails the they will have the same dynamics and behavior.
The similarity above requires that the transformation matrix be the $\mathbf{I}$ unity matrix in appropriate dimension. That is, $\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} = \mathbf{I}^{-1}\big(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t)))\mathbf{C}\big)\mathbf{I}$, thus $\mathbf{G}(\mathbf{p}(t))$ can be calculated on the Ω parameter domain at every $\mathbf{p}(t)$.

$$\begin{aligned}& \mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} = \mathbf{I}^{-1}\big(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t)))\mathbf{C}\big)\mathbf{I} = \\ & = \mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(\mathbf{p}(t)))\mathbf{C} \\ & \mathbf{G}(\mathbf{p}(t)) = -(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C} - \mathbf{A}(\mathbf{p}(t)) + \mathbf{G}_{ref}\mathbf{C})\mathbf{C}^{-1} = -(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1}\end{aligned} \quad . \tag{11}$$

Therefore, the (10) should be modified accordingly to (11) which leads back to (4):

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{G}(\mathbf{p}(t))\mathbf{C}\big)\hat{\mathbf{x}}(t) + \mathbf{G}(\mathbf{p}(t))\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) = \\ &= \big(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} + \mathbf{G}(t))\mathbf{C}\big)\hat{\mathbf{x}}(t) + (\mathbf{G}_{ref} + \mathbf{G}(t))\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) = \\ &= \big(\mathbf{A}(\mathbf{p}(t)) - (\mathbf{G}_{ref} - (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{C}\big)\hat{\mathbf{x}}(t) + \\ &+ (\mathbf{G}_{ref} - (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) = \\ &= \big(\mathbf{A}(\mathbf{p}(t)) - \mathbf{G}_{ref}\mathbf{C} + \mathbf{A}_{ref}\mathbf{C}^{-1}\mathbf{C} - \mathbf{A}(\mathbf{p}(t))\mathbf{C}^{-1}\mathbf{C}\big)\hat{\mathbf{x}}(t) + \\ &+ (\mathbf{G}_{ref} - (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) = \\ &= \big(\mathbf{A}_{ref} - \mathbf{G}_{ref}\mathbf{C}\big)\hat{\mathbf{x}}(t) + (\mathbf{G}_{ref} - (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t)\end{aligned} \quad . \tag{12}$$

2.6 Consequences and limitations

As it was declared above, the $p_i(t)$ scheduling parameters has to be measured or estimated since these are directly used for tuning the developed controller and observer structures. The main limitations are the invertibility of **B** and **C** matrices – which are key properties regarding the applicability. Thus, if **B** and **C** are fully invertable, then (9) and (12) can be applied and complementary LPV controller and observer design is possible.
However, there are generalization possibilities based on Definition 4 - 7 which can be utilized in order to make the developed solutions more flexible.

2.6.1 Controller Side

1. *Appropriate selection of $p_i(t)$ scheduling parameters.*
It is possible to calculate the $\mathbf{K}(\mathbf{p}(t))$ matrix in element by element way without inversion of **B**. The key components are the selection and linking of $p_i(t)$.

Remark 5. The expression "linking" should be explained at this point. If we select a nonlinearity inducing term from a given equation, we can "link" it to a given state variable in a natural or forced way depending on the structure of the equation and the requirements detailed below. For example, assume the following simple equations:

$$\begin{aligned}&\dot{x}_1(t) = k_1x_1(t)x_2(t) + k_2\sqrt{x_2(t)} + k_3x_2(t) \\ &\dot{x}_2(t) = -k_2\sqrt{x_2(t)} - k_3x_2(t) + u_1(t)\end{aligned} \quad . \tag{13}$$

In (13) two nonlinearity inducing elements can be found, $x_1(t)x_2(t)$ and $\sqrt{x_2(t)}$.
Natural linking: we can select $p_1(t) = k_1x_2(t)$, which means we link $p_1(t)$ to $x_1(t)$ in this equation as $\dot{x}_1(t) = p_1(t)x_1(t) + k_2\sqrt{x_2(t)} + k_3x_2(t)$. This linking is a natural choice and come from the structure of the equation.
Forced linking: we have to select $p_2(t) = k_2\sqrt{x_2(t)}$ and link to a state. It is possible by using simple manipulations, eg. multiplication by $1 = \frac{x_1(t)}{x_1(t)}$. Therefore, $k_2\sqrt{x_2(t)}\frac{x_1(t)}{x_1(t)}$ occurs and $p_2(t) = k_2\frac{\sqrt{x_2(t)}}{x_1(t)}$ will be the selected scheduling variable. Thus, $p_2(t)$ can be linked in a forced way to $x_1(t)$ as $\dot{x}_1(t) = p_1(t)x_1(t) +$

$p_2(t)x_1(t) + k_3x_2(t)$. Strong limitation is that $x_1(t) \neq 0 \ \forall t \geq 0$ in order to avoid singularity. With forced linking we can arbitrarily bound given terms as scheduling variables to selected states beside the mentioned limitation.
It has to be mentioned that in case of the forced linking, we have to be sure that denominator of the newly realized scheduling variable – the state to which the scheduling variable is linked – not only cannot be zero during operation. However, it also has to be measurable, since we use it as an external "input". The other solution is to estimate this state via nonlinear state estimators (such as the Kalman filter [12]).

From the control input side, the structure of **B** determines the selection and linking of $p_i(t)$. From (7) it is clear that $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ difference matrix does only contain elements in its structure where scheduling variables can be found – since the other elements of the matrices are the same and by the $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ subtraction become zero. However, in those entries which contain scheduling variables $p_{i,j,ref} - p_{i,j}(t)$ difference occur.
The structure of **B** will be also important. Suppose that every column and row of **B** does contain at most one non zero element regardless the position (entry) of the element in the structure of **B**. That means that every state could have at most one different control input – which is reasonable in most of the physical and physiological systems. For example, in case of a system with three states which have three inputs, **B** can only contain one elements in each row, e.g..: $\mathbf{B} = \begin{bmatrix} 0 & b_2 & 0 \\ b_1 & 0 & 0 \\ 0 & 0 & b_3 \end{bmatrix}$.
Assume that the structure of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ is such that it contains $p_{i,j,ref} - p_{i,j}(t)$ elements in only those rows where the rows of **B** does have non zero $b_{i,j}$ elements and the previous statement for **B** is true (columns and rows regardless the position does contain only one element). In this case, the elements of $\mathbf{K}(t)$, namely $k_{i,j}(t)$ can be calculated in an inverse way from the corresponding $p_{i,j,ref} - p_{i,j}(t)$ and $b_{i,j}$. Thus, we know that $p_{i,j,ref} - p_{i,j}(t) = b_{i,j}k_{i,j}(t) \rightarrow k_{i,j}(t) = \dfrac{p_{i,j,ref} - p_{i,j}(t)}{b_{i,j}}$.
The last missing piece in this regard is to establish the equality of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{BK}(t)$, which is true when $\mathrm{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = \mathrm{rank}(\mathbf{BK}(t))$. This rank criteria can be covered by the Definitions 4 - 7.
Assume that $\mathbf{BK}(t)$ can be decomposed to a dyad as $\mathbf{BK}(t) = \sum_{i=1}^{g} \mathbf{b}_i \mathbf{k}(\mathbf{p}(t))_i^\top$ and $\sum_{i=1}^{g} \mathbf{b}_i \mathbf{k}(\mathbf{p}(t))_i^\top$ is a minimal dyadic decomposition of $\mathbf{BK}(t)$. In this case, $\mathrm{rank}(\mathbf{BK}(t)) = g$. Having regard to this fact we have to select the scheduling variables in such a way that $\mathrm{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = g$ as well. It is only possible, if the structure of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ does contain g linearly independent columns (or rows). Then, the rank criteria automatically satisfies and $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \sum_{i=1}^{g} \mathbf{b}_i \mathbf{k}(\mathbf{p}(t))_i^\top$ which means that $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ can be described as the sum of g piece of dyadic products.
However, the number of restrictions seems high, but in practice most of them is automatically satisfied and with forced linking we can link the scheduling variables to

a given state arbitrarily.
For example, in case of a system with three states, two control inputs and two scheduling variables:

$$\mathbf{B} = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \end{bmatrix}, \;\; \mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} p_{1,ref} - p_1(t) & 0 & 0 \\ 0 & p_{2,ref} - p_2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \sum_{i=1}^{2} \mathbf{b}_i \mathbf{k}(\mathbf{p}(t))_i^\top = \begin{bmatrix} p_{1,ref} - p_1(t) & 0 & 0 \\ 0 & p_{2,ref} - p_2(t) & 0 \\ 0 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} b_1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} \frac{p_{1,ref} - p_1(t)}{b_1} & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ b_2 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{p_{2,ref} - p_2(t)}{b_2} & 0 \end{bmatrix} \tag{14}$$

$$\mathbf{K}(t) = \begin{bmatrix} \frac{p_{1,ref} - p_1(t)}{b_1} & 0 & 0 \\ 0 & \frac{p_{2,ref} - p_2(t)}{b_2} & 0 \end{bmatrix}$$

$$\text{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = 2$$

2. *Control input virtualization.*
There are special opportunities to "virtually" increase the number of control input signals, if there is only one control input. Thus, the structure of **B** can be extended with additional columns with appropriate entries. From the control input side, it means that all state equations can be completed additionally with $u_{virt,i}(t)$ "virtual" inputs via the duplication of the real control input. In this case, the $u_{virt,i}(t)$ virtual input signals have to be equal to the real control input, namely $u_{virt,i}(t) = u_{real}(t)$ regardless of how many $u_{virt,i}(t)$ virtual inputs are considered. The main restriction will be that all of the rows of the realized $\mathbf{K}(t)$ have to be equal, which is the only way to reach the equality of $u_{virt,i}(t) = u_{real}(t)$. The usage of this technique requires the assumptions from the previous section regarding the structure of **B** and $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$.
The input virtualization technique will be introduced via a practical example. Assume a three state system – with $x_1(t)$, $x_2(t)$ and $x_3(t)$ states – which contains a control input signal in its third state equation and a selected scheduling variable can be found only in the third equation linked to the first state as follows:

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + a_2 x_2(t) \\ \dot{x}_2(t) &= -a_2 x_2(t) + a_3 x_3(t) \\ \dot{x}_3(t) &= -x_1(t)\sqrt{x_3(t)} - a_3 x_3(t) + b_1 u_{real}(t) \end{aligned} \;, \tag{15}$$

where $p_1(t) = -\sqrt{x_3(t)}$ is selected as scheduling variable. In (15) $\mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ b_1 \end{bmatrix}$ and

$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_1(t) & 0 & 0 \end{bmatrix}$. Introduce two new virtual inputs into the previous equation:

$$\begin{array}{l} \dot{x}_1(t) = -a_1 x_1(t) + a_2 x_2(t) + c_1 u_{virt,1}(t) - c_1 u_{virt,1}(t) \\ \dot{x}_2(t) = -a_2 x_2(t) + a_3 x_3(t) + c_2 u_{virt,2}(t) - c_2 u_{virt,2}(t) \\ \dot{x}_3(t) = p_1(t) x_1(t) - a_3 x_3(t) + b_1 u_{real}(t) \end{array}, \tag{16}$$

In this case, an extended input matrix can be introduced: $\mathbf{B}_{ex} = \begin{bmatrix} c_1 b_1 & -c_1 b_1 & 0 \\ c_2 b_1 & 0 & -c_2 b_1 \\ b_1 & 0 & 0 \end{bmatrix}$, where $c_1 := 1[\dot{x}_1(t)/\dot{x}_3(t)]$ and $c_2 := 1[\dot{x}_2(t)/\dot{x}_3(t)]$ converter scalers take care of the appropriate units and $u_{virt,1}(t) = u_{virt,2}(t) = u_{real}(t)$. Since the concrete values of c_1 and c_2 are equal to 1, they will not be indicated in the followings.The extended $\mathbf{B}_{ex}$ is invertible, namely $\mathbf{B}_{ex}^{-1} = \begin{bmatrix} 0 & 0 & b_1 \\ -b_1 & 0 & b_1 \\ 0 & -b_1 & b_1 \end{bmatrix}$. In this case, $\mathbf{K}(t)$ can be calculated by using (7) such as:

$$\begin{array}{l} \mathbf{K}(t) = -\mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = \\ = -\begin{bmatrix} 0 & 0 & b_1 \\ -b_1 & 0 & b_1 \\ 0 & -b_1 & b_1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ p_{1,ref} - p_1(t) & 0 & 0 \end{bmatrix} = \\ = -\begin{bmatrix} b_1(p_{1,ref} - p_1(t)) & 0 & 0 \\ b_1(p_{1,ref} - p_1(t)) & 0 & 0 \\ b_1(p_{1,ref} - p_1(t)) & 0 & 0 \end{bmatrix} \end{array}. \tag{17}$$

The mentioned key component is that $u_{virt,i}(t) = u_{real}(t)$ and the configuration of (15) will provide this restriction.
In general, the states feedback design does not modify in case of the reference LTI system, namely, the state feedback designing process have to be done by using the original $\mathbf{B} = \begin{bmatrix} 0 & 0 & b_1 \end{bmatrix}^\top$. The $\mathbf{K}_{ref}$ feedback gain will be a row matrix as $\mathbf{K}_{ref} = \begin{bmatrix} k_{1,ref} & k_{2,ref} & k_{3,ref} \end{bmatrix}$. In this given case to reach $u_{virt,i}(t) = u_{real}(t)$, we have to duplicate the rows of $\mathbf{K}_{ref}$ and realize an extended feedback gain matrix, such as $\mathbf{K}_{ref,ex} = \begin{bmatrix} k_{1,ref} & k_{2,ref} & k_{3,ref} \\ k_{1,ref} & k_{2,ref} & k_{3,ref} \\ k_{1,ref} & k_{2,ref} & k_{3,ref} \end{bmatrix}$. By using the extended $\mathbf{B}_{ex}$ in the control law description the virtual inputs will drop out from the given state equations and will be represented as an addition of zero in these (e.g. $+0 := +u_{virt} - u_{virt}$), which is a

realizable configuration by state feedback.

$$\dot{\mathbf{x}}(t) = \mathbf{A}(\mathbf{p}(t))\mathbf{x}(t) + \mathbf{B}_{ex}\mathbf{u}_{ex}(t) =$$
$$= \begin{bmatrix} -a_1 & a_2 & 0 \\ 0 & -a_2 & a_3 \\ p_1(t) & 0 & -a_3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} b_1 & -b_1 & 0 \\ b_1 & 0 & -b_1 \\ b_1 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{virt,1}(t) \\ u_{virt,2}(t) \\ u_{real}(t) \end{bmatrix} =$$
$$= (\mathbf{A}(\mathbf{p}(t)) - \mathbf{B}_{ex}(\mathbf{K}_{ref} + \mathbf{K}(t)))\mathbf{x}(t) =$$
$$= \left(\begin{bmatrix} -a_1 & a_2 & 0 \\ 0 & -a_2 & a_3 \\ p_1(t) & 0 & -a_3 \end{bmatrix} - \begin{bmatrix} b_1 & -b_1 & 0 \\ b_1 & 0 & -b_1 \\ b_1 & 0 & 0 \end{bmatrix} \right.$$
$$\left. \left(\begin{bmatrix} k_{1,ref} & k_{2,ref} & k_{3,ref} \\ k_{1,ref} & k_{2,ref} & k_{3,ref} \\ k_{1,ref} & k_{2,ref} & k_{3,ref} \end{bmatrix} - \begin{bmatrix} b_1(p_{1,ref} - p_1(t)) & 0 & 0 \\ b_1(p_{1,ref} - p_1(t)) & 0 & 0 \\ b_1(p_{1,ref} - p_1(t)) & 0 & 0 \end{bmatrix} \right) \right) \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} =$$
$$= \begin{bmatrix} -a_1 & a_2 & 0 \\ 0 & -a_2 & a_3 \\ p_1(t) - b_1(k_{1,ref} - b_1(p_{1,ref} - p_1(t))) & 0 - b_1 k_{2,ref} & -a_3 - b_1 k_{3,ref} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} \tag{18}$$

From (18) it is clear that the input virtualization does not modify the first and second state equation via the state feedback, however, directly affects the third equation in which the control input occurs. At the same time, this construction provides the restriction from above in general (eigenvalue equality, rank criteria, etc.).
Naturally, other constructions can be imagined, but each case requires unique construction of $\mathbf{B}_{ex}$ and careful selection of $p_i(t)$. Regarding the given case, the same result occurs if all of the states have linked scheduling parameters in the third state equation (beside keeping in mind the limitations of selection of them). Namely, $\mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p_1(t) + \text{AD} & p_2(t) + \text{AD} & p_3(t) + \text{AD} \end{bmatrix}$, where AD means those additional coefficients of the states which are not embedded into a scheduling variable. Moreover, the difference matrix becomes

$$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ p_{1,ref} - p_1(t) & p_{2,ref} - p_2(t) & p_{3,ref} - p_3(t) \end{bmatrix} . \tag{19}$$

It can be observed that the $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ difference matrix contains elements only in the third row which corresponds to that row in $\mathbf{B}$ which contains the real input coefficient.

Remark 6. It has to be noticed that the mentioned techniques which help to get around the invertibility issue of $\mathbf{B}$ strongly coupled to the complementary observer design. The structure of **C** is similarly important and determines the usage of same techniques regarding the observer design.

In multi input case the input virtualization technique may not be applicable. It depends on the structure of **B** and **C** matrices – however, further generalization from this point of view is ongoing.

2.6.2 Observer Side

From (11) it is clear that the key point is the invertibility of **C**. This is only possible if all of the states are represented in the output, so directly measurable or calculable. Otherwise, if **C** is not invertible, we have to face the same problem as in case of the invertiblity of **B**. Although the same solution – appropriate selection of $p_i(t)$ from the output point of view – can be used for element by element calculation of $\mathbf{G}(t)$ observer gain. Virtualization of the output is meaningless from the output point of view.

The structure of **C** determines the selection and linking of $p_i(t)$. Equation (11) shows that $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ difference matrix does only contain elements in those entries where scheduling variables can be found which are equal to $p_{i,j,ref} - p_{i,j}(t)$ difference.

The other component to be investigated is the structure of **C**. Assume that every rows and columns of **C** contain at most one non zero element regardless the position (entry) of the element in the structure of **C**. From system point of view this assumption is reasonable, since in most of the physical or physiological systems each output connects to one state. For example, in case of a system with three states which have two outputs connected to $x_1(t)$ and $x_2(t)$ states, **C** can only be $\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \end{bmatrix}$. Assume that the structure of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ such that it contains $p_{i,j,ref} - p_{i,j}(t)$ elements in only those columns where the columns of **C** does have non zero $c_{i,j}$ elements and the previous statement for **C** is true (columns and rows regardless the position does contain only one element). In this case, the elements of $\mathbf{G}(t)$, namely $g_{i,j}(t)$ can be calculated in the same inverse way as $\mathbf{K}(t)$ from the corresponding $p_{i,j,ref} - p_{i,j}(t)$ and $c_{i,j}$. Thus, we know that $p_{i,j,ref} - p_{i,j}(t) = g_{i,j}(t)c_{i,j} \rightarrow g_{i,j}(t) = \dfrac{p_{i,j,ref} - p_{i,j}(t)}{c_{i,j}}$.

The equality $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{G}(t)\mathbf{C}$ holds if $\mathrm{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = \mathrm{rank}(\mathbf{G}(t)\mathbf{C})$. Again, by using the consequences of Definitions 4 - 7 this rank criteria can be proven.

Assume that $\mathbf{G}(t)\mathbf{C}$ can be decomposed to a dyad as $\mathbf{G}(t)\mathbf{C} = \sum_{i=1}^{f} \mathbf{g}(\mathbf{p}(t))_i \mathbf{c}_i^\top$ and $\sum_{i=1}^{f} \mathbf{g}(\mathbf{p}(t))_i \mathbf{c}_i^\top$ is a minimal dyadic decomposition of $\mathbf{G}(t)\mathbf{C}$. In this case $\mathrm{rank}(\mathbf{G}(t)\mathbf{C}) = f$. Due to this fact, we have to select the scheduling variables in such a way that $\mathrm{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))) = f$ as well. It is only possible if the structure of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ does contain f linearly independent columns (or rows). Then the rank cri-

teria is automatically satisfied and $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))=\sum_{i=1}^{f}\mathbf{g}(\mathbf{p}(t))_i\mathbf{c}_i^\top$ which means that $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))$ can be described as the sum of f piece of dyadic products.
Similar to the previous case by forced linking of the scheduling variables and simple mathematical manipulations it can be achieved that the described conditions are automatically satisfied in practice.
For example, in case of a system with three states, two outputs and two scheduling variables:

$$\mathbf{C}=\begin{bmatrix}c_1 & 0 & 0\\ 0 & c_2 & 0\end{bmatrix},\ \mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))=\begin{bmatrix}p_{1,ref}-p_1(t) & 0 & 0\\ 0 & p_{2,ref}-p_2(t) & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))=\sum_{i=1}^{2}\mathbf{g}(\mathbf{p}(t))_i\mathbf{c}_i^\top=\begin{bmatrix}p_{1,ref}-p_1(t) & 0 & 0\\ 0 & p_{2,ref}-p_2(t) & 0\\ 0 & 0 & 0\end{bmatrix}=$$

$$=\begin{bmatrix}\frac{p_{1,ref}-p_1(t)}{c_1}\\ 0\\ 0\end{bmatrix}\begin{bmatrix}c_1 & 0 & 0\end{bmatrix}+\begin{bmatrix}0\\ \frac{p_{2,ref}-p_2(t)}{c_2}\\ 0\end{bmatrix}\begin{bmatrix}0 & c_2 & 0\end{bmatrix} \tag{20}$$

$$\mathbf{G}(t)=\begin{bmatrix}\frac{p_{1,ref}-p_1(t)}{c_1} & 0\\ 0 & \frac{p_{2,ref}-p_2(t)}{c_2}\\ 0 & 0\end{bmatrix}$$

$$\mathrm{rank}(\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t)))=2$$

2.6.3 Connection between the Controller and Observer side

If the **B** and **C** is ivertible then the calculated $\mathbf{K}(t)$ and $\mathbf{G}(t)$ practically separated from each other.
By using the mentioned element-by-element calculation, the connection between the $\mathbf{K}(t)$ and $\mathbf{G}(t)$, furthermore between **B** and **C** is straightforward. The structure of $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))$ have to satisfy the requirements defined by the structures of **B** and **C**. Namely, non zero elements can be in only those rows of $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))$ where the corresponding rows of **B** have non zero elements. Moreover, non zero elements can be in only those columns of $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))$ where the corresponding columns of **C** have non zero elements. Furthermore, the measurability of $\mathbf{p}(t)$ parameter vector has to be kept in mind all the time. If $\mathbf{p}(t)$ cannot be measured directly, estimation of it is needed, for example via Kalman filtering.
From realization point of view this means that the complementary controller and observer design have to be investigated in a strong conjunction and forced linking should be applied in order to reach the appropriate structure for $\mathbf{A}_{ref}-\mathbf{A}(\mathbf{p}(t))$ to

find the trade-off between the control and observer requirements come from the **B** and **C**.
To provide a full picture, the following practical example demonstrates this balancing between the requirements. Assume a given four state system with two inputs (connected to $x_3(t)$ and $x_4(t)$) and two outputs (connected to $x_1(t)$ and $x_3(t)$). First, we have to investigate where the $p_{i,ref} - p_i(t)$ differences can occur in the $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ embedded in the system matrix to test the applicability of the method. In this case, the investigation will be extended to the controller and observer parts as well from **B** and **C** point of view. Be $\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_1 & 0 \\ 0 & b_2 \end{bmatrix}$.
Denote the entries of $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ with $\Delta a_{i,j}(t)$. According to the prescriptions regarding **B** and **C** non zero $\Delta p_i(t)$ elements in $\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))$ can be occurred only in $\Delta a_{3,1}(t), \Delta a_{3,3}(t), \Delta a_{4,1}(t)$ and $\Delta a_{4,3}(t)$. At this given case for calculate $\mathbf{K}(t)$ and $\mathbf{G}(t)$ the following equations can be written:

$$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{BK}(t) \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ b_1 & 0 \\ 0 & b_2 \end{bmatrix} \begin{bmatrix} \frac{\Delta a_{3,1}(t)}{b_1} & 0 & \frac{\Delta a_{3,3}(t)}{b_2} & 0 \\ \frac{\Delta a_{4,1}(t)}{b_1} & 0 & \frac{\Delta a_{4,3}(t)}{b_2} & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta a_{3,1}(t) & 0 & \Delta a_{3,3}(t) & 0 \\ \Delta a_{4,1}(t) & 0 & \Delta a_{4,3}(t) & 0 \end{bmatrix}$$

$$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{G}(t)\mathbf{C} \rightarrow \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\Delta a_{3,1}(t)}{c_1} & \frac{\Delta a_{3,3}(t)}{c_2} \\ \frac{\Delta a_{4,1}(t)}{c_1} & \frac{\Delta a_{4,3}(t)}{c_2} \end{bmatrix} \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \Delta a_{3,1}(t) & 0 & \Delta a_{3,3}(t) & 0 \\ \Delta a_{4,1}(t) & 0 & \Delta a_{4,3}(t) & 0 \end{bmatrix} \tag{21}$$

where the complementary feedback and observer gains can be calculated as

$$\mathbf{K}(t) = \begin{bmatrix} \frac{\Delta a_{3,1}(t)}{b_1} & 0 & \frac{\Delta a_{3,3}(t)}{b_2} & 0 \\ \frac{\Delta a_{4,1}(t)}{b_1} & 0 & \frac{\Delta a_{4,3}(t)}{b_2} & 0 \end{bmatrix}, \quad \mathbf{G}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{\Delta a_{3,1}(t)}{c_1} & \frac{\Delta a_{3,3}(t)}{c_2} \\ \frac{\Delta a_{4,1}(t)}{c_1} & \frac{\Delta a_{4,3}(t)}{c_2} \end{bmatrix}. \tag{22}$$

If the aforementioned restrictions and requirements are held for the calculation of the gains, then the connection between them is obvious from (20) and (21):

$$\begin{aligned} &\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{B}\mathbf{K}(t) = \mathbf{G}(t)\mathbf{C} \\ &\text{rank}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \text{rank}(\mathbf{B}\mathbf{K}(t)) = \text{rank}(\mathbf{G}(t)\mathbf{C}) \end{aligned} . \quad (23)$$

The suggested element-by-element calculation cannot be used all the time – it depends on the given system to be controlled and usability requires deep investigation of the possible LPV structures of the system.

2.7 Feed Forward Compensator

Due to the basic properties of classical state feedback control an additional feed forward compensator has to be embedded in the control loop. Without it the state feedback controller enforces the states (and though the outputs) to reach zero values over time during operation. In order to reach the desired steady state values of the output this $\mathbf{N}(\mathbf{p}(t)) = \begin{pmatrix} \mathbf{N}_x(\mathbf{p}(t)) \\ \mathbf{N}_u(\mathbf{p}(t)) \end{pmatrix}$ feed forward compensator should be $\mathbf{p}(t)$-dependent as well [13, 19, 20].
The $\mathbf{p}(t)$ dependent compensator matrices can be calculated as follows [13, 4]:

$$\begin{aligned} &\begin{bmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix} \begin{bmatrix} \mathbf{N}_x(\mathbf{p}(t)) \\ \mathbf{N}_u(\mathbf{p}(t)) \end{bmatrix} = \begin{bmatrix} \mathbf{0}_{n\times m} \\ \mathbf{I}_m \end{bmatrix} \\ &\begin{bmatrix} \mathbf{N}_x(\mathbf{p}(t)) \\ \mathbf{N}_u(\mathbf{p}(t)) \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\mathbf{p}(t)) & \mathbf{B} \\ \mathbf{I}_n & \mathbf{0}_{n\times m} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0}_{n\times m} \\ \mathbf{I}_m \end{bmatrix} \end{aligned} , \quad (24)$$

where $\mathbf{I}_n$ is the feedback "selector" matrix (here is a unity matrix), $\mathbf{0}_{n\times m}$ is zero matrix and $\mathbf{I}_m$ is unity matrix.
The compensator does modify the state vector by substracting the desired steady-state from the actual state of the system, the steady-state is calculated as $\mathbf{N}_x(\mathbf{p}(t))r(t)$, where $r(t)$ is the reference signal in the time instant t, and it modifies the control input by adding the steady-state control input calculated as $\mathbf{N}_u(\mathbf{p}(t))r(t)$. Therefore, the controller is governed by the equations:

$$\begin{aligned} &\dot{\hat{\mathbf{x}}}(t) = \mathbf{F}\hat{\mathbf{x}}(t) + (\mathbf{G}_{ref} + (\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)))\mathbf{C}^{-1})\mathbf{y}(t) + \mathbf{H}\mathbf{u}(t) \\ &\mathbf{u}(t) = (\mathbf{K}_{ref} + \mathbf{B}^{-1}(\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t))))(\hat{\mathbf{x}}(t) - \mathbf{N}_x(\mathbf{p}(t))\mathbf{r}(t)) + \mathbf{N}_u(\mathbf{p}(t))\mathbf{r}(t) \end{aligned} . \quad (25)$$

2.8 Particular steps to realize complementary LPV controller and observer in practice

Here we have collected the main steps of the realization of the complementary LPV controller and observer structure.

- Realization of the appropriate $\mathbf{S}(\mathbf{p}(t))$ LPV model form from the original non-linear model,

- Selection of the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system (which is an underlying LTI system as well),
- Design of the $\mathbf{K}_{ref}$ reference state feedback controller with an arbitrary method, which is appropriate to handle the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system,
- Design of the $\mathbf{G}_{ref}$ reference observer gain with an arbitrary method which is appropriate to observes the $\mathbf{S}(\mathbf{p}_{ref})$ reference LTI system,
- Realization of the complementary LPV controller and observer structure based on Fig. 1.

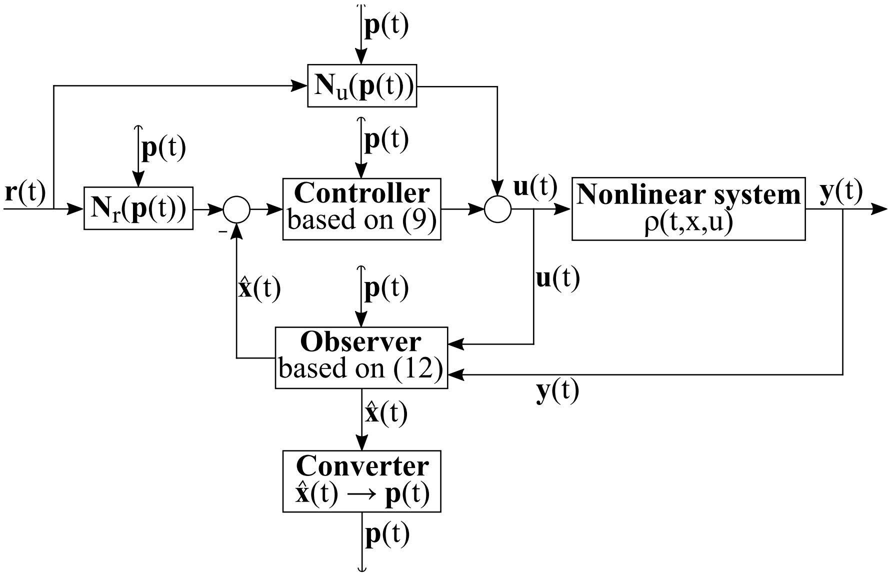

Figure 1
General feedback control loop with completed controller and observer.

3 Control of Innate Immune Response

In this section a spectacular physiological control example will be demonstrated by using the aforementioned methods.
Control of the response of innate immune system for given loads is crucial in many cases these days, especially when the patient's quality of life depend on it. Organ transplantation as final solution in case of organ disorders and malfunctions requires strict immunosuppression in order to prevent the rejection of the transplanted organ [21]. Furthermore, in case of many autoimmune diseases the effective immunosuppression is the only way to avoid the self-destruction of the human body by automated mechanisms [22]. On the other hand, suppression of internal defense system could lead ot unwanted states, e.g. activates of carried, but inactive viruses or bacteria and less effective immune protection against cancer [23, 24]. For example, the resting cytomegalovirus infection – which does not cause problems for a

healthy human – may causes serious problems for people with transplanted kidney or liver. The virus, even if it comes together with a donor organ or belongs to the recipient may lead to massive inflammation and critical state of different organs, if the immune suppression is strong [21].
Therefore, the accurate description of the immune response by mathematical models which can be basis for control design is very useful in these cases. In the following, we show a general model which can be adopted for various instances in order to describe the dynamics of infections and the response of the immune system for that.

3.1 Applied Model

The mathematical description of the used general theoretical model appeared in [25]. This model has beneficial properties, because it is able to describe the dynamics of several different diseases and its structure can be dynamically transformed in order adapt to the particular disease to be modeled and/or controlled.
$x_1(t)$ is the concentration of a pathogen, however, this can be measured by the concentration of associated antigen, $x_2(t)$ is the concentration of plasma the cells carrying and producing the antibodies, $x_3(t)$ is the concentration of antibodies which kill the pathogen and $x_4(t)$ is the relative characteristics of the damaged organ (where $x_4 = 0$ and $x_4 = 1$ mean the "healthy" and "dead" conditions, respectively). In general, the values of the states cannot be lower than zero ($x_i(t) \geq 0$, $i = [1,2,3,4]$, $\forall t \geq 0$).
An important property of the model has to be highlighted, namely, the $x_i(t)$ states are concentrations, however, the concrete units are not given due to the model is general in the given form and can be adopted to a wide range of cases. The same is true for the time span as well, namely, it can be arbitrarily determined to make the applicability of the model more flexible. Because of these facts, in this study the concrete type of concentration and time span are handled as "general units", without specification – similarly as [25].
The model consists of the following ordinary and delayed differential equations:

$$\dot{x}_1(t) = (a_{11} - a_{12}x_3(t))x_1(t) + b_1u_1(t), \tag{26a}$$

$$\dot{x}_2(t) = a_{21}(x_4(t))a_{22}x_1(t-\tau)x_3(t-\tau) - a_{23}(x_2(t) - x_2^*) + b_2u_2(t), \tag{26b}$$

$$\dot{x}_3(t) = a_{31}x_2(t) - (a_{32} + a_{33}x_1(t))x_3(t) + b_3u_3(t), \tag{26c}$$

$$\dot{x}_4(t) = a_{41}x_1(t) - a_{42}x_4(t) + b_4u_4(t), \tag{26d}$$

where $a_{11} = 1$, $a_{12} = 1$, $a_{22} = 3$, $a_{23} = 1$, $a_{31} = 1$, $a_{32} = 1.5$, $a_{33} = 0.5$, $a_{41} = 1$, $a_{42} = 1$, $b_1 = -1$, $b_2 = 1$, $b_3 = 1$ and $b_4 = -1$ are constant parameters of the model. In this study, the applied values of the parameters were the same as in [25]. The model does contain a saturation as follows:

$$a_{21}(x_4(t)) = \begin{cases} cos\pi x_4(t), & 0 \leq x_4(t) \leq 0.5 \\ 0, & \text{otherwise} \end{cases} . \tag{27}$$

In this case – similarly to [25] – the τ constant time delays are not taken into consideration in states $x_1(t)$ and $x_3(t)$.
In order to highlight what are the critical parts which shall be handled as scheduling variables (25) can be described in extended and completed form.

$$\dot{x}_1(t) = (a_{11} - a_{12}x_3(t))x_1(t) + b_1u_1(t) = p_1(t)x_1(t) + b_1u_1(t), \tag{28a}$$

$$\begin{aligned}\dot{x}_2(t) = \; & a_{21}(x_4(t))a_{22}x_1(t)x_3(t) - a_{23}(x_2(t) - x_2^*) + b_2u_2(t) = \\ & a_{21}(x_4(t))a_{22}x_1(t)x_3(t) - a_{23}x_2(t) + a_{23}x_2^*1 + b_2u_2(t) = \\ & a_{21}(x_4(t))a_{22}x_3(t)x_1(t) - a_{23}x_2(t) + \frac{a_{23}x_2^*}{x_3(t)}x_3(t) + b_2u_2(t) = \\ & p_2(t)x_1(t) - a_{23}x_2(t) + p_3(t)x_3(t) + b_2u_2(t)\end{aligned} \tag{28b}$$

$$\begin{aligned}\dot{x}_3(t) = \; & a_{31}x_2(t) - a_{32}x_3(t) - a_{33}x_3(t)x_1(t) + b_3u_3(t) = \\ & a_{31}x_2(t) - a_{32}x_3(t) + p_4(t)x_1(t) + b_3u_3(t),\end{aligned} \tag{28c}$$

$$\dot{x}_4(t) = a_{41}x_1(t) - a_{42}x_4(t) + b_4u_4(t), \tag{28d}$$

where $p_1(t) = a_{11} - a_{12}x_3(t)$, $p_2(t) = a_{21}(x_4(t))a_{22}x_3(t)$, $p_3(t) = \frac{a_{23}x_2^*}{x_3(t)}$ and $p_4(t) = -a_{33}x_3(t)$ are the selected scheduling variables, respectively. Hence, the parameter vector becomes $\mathbf{p}(t) = [p_1(t), p_2(t), p_3(t), p_4(t)]^\top$. Therefore, a $4D$ parameter space occurs.

The outputs of such a theoretical system is not predefined, but also depend on the given application. In this study, the followings are considered: $x_1(t)$, $x_3(t)$ and $x_4(t)$ are selected as outputs, namely these are measurable. The concentration of possible pathogens are usually higher than the concentration of associated antigens [26, 21] – this is taken into account with a scaler c_1 at the output side of $x_1(t)$ – therefore, $c_1x_1(t)$ term is handled as measurable outputs. In this way, the outputs of the system are $y_1(t) = c_1x_1(t)$, $y_2(t) = c_2x_3(t)$ and $y_3(t) = c_3x_4(t)$ where $c_1 = 1.5$, $c_2 = 1$ and $c_3 = 1$, respectively. Now, the system matrices of the LPV system arises as follows:

$$\mathbf{A}(\mathbf{p}(t)) = \begin{bmatrix} p_1(t) & 0 & 0 & 0 \\ p_2(t) & -a_{23} & p_3(t) & 0 \\ p_4(t) & a_{31} & -a_{32} & 0 \\ a_{41} & 0 & 0 & -a_{42} \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} b_1 & 0 & 0 & 0 \\ 0 & b_2 & 0 & 0 \\ 0 & 0 & b_3 & 0 \\ 0 & 0 & 0 & b_4 \end{bmatrix}.$$
$$\mathbf{C} = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & 0 & c_2 & 0 \\ 0 & 0 & 0 & c_3 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{29}$$

The LPV system can be written in compact form:

$$\begin{pmatrix}\dot{\mathbf{x}}(t)\\ \mathbf{y}(t)\end{pmatrix} = \mathbf{S}(\mathbf{p}(t))\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} = \begin{bmatrix}\mathbf{A}(\mathbf{p})(t) & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} =$$
$$= \begin{bmatrix} p_1(t) & 0 & 0 & 0 & b_1 & 0 & 0 & 0\\ p_2(t) & -a_{23} & p_3(t) & 0 & 0 & b_2 & 0 & 0\\ p_4(t) & a_{31} & -a_{32} & 0 & 0 & 0 & b_3 & 0\\ a_{41} & 0 & 0 & -a_{42} & 0 & 0 & 0 & b_4\\ c_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & c_2 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & c_3 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix}\mathbf{x}(t)\\ \mathbf{u}(t)\end{pmatrix} . \quad (30)$$

3.2 Design of the Complementary LPV Controller

The first step is to determine the reference LTI model. Assume that a set of favorable states is $\mathbf{x}_{favorable} = [0, x_2^*, (2x_2^*/3), 0]^\top$ which describes healthy condition without presence of pathogens [25]. Through $\mathbf{x}_{favorable}$ the $\mathbf{p}_{ref} = [-0.3333, 0, 1.5, -0.6667]$ can be used. Moreover, the $\mathbf{r}$ reference vector can be selected to be equal to $\mathbf{x}_{favorable}$, as the desired values which have to be reached by the states over time ($\mathbf{r} = \mathbf{x}_{favorable} = [0, x_2^*, (2x_2^*/3), 0]^\top$ over $t \to \infty$). The $\mathbf{A}(\mathbf{p}_{ref})$ becomes

$$\mathbf{A}(\mathbf{p}_{ref}) = \begin{bmatrix} -0.3333 & 0 & 0 & 0\\ 0 & -1 & 1.5 & 0\\ -0.6667 & 1 & -1.5 & 0\\ 1 & 0 & 0 & -1 \end{bmatrix} . \quad (31)$$

The eigenvalues of the $\mathbf{A}(\mathbf{p}_{ref})$ are $\lambda(\mathbf{A}(\mathbf{p}_{ref})) = [-2.5, 0, -1, -0.3333]^\top$, so the reference LTI system is close to the border of stability because of its pole at 0.
The rank of the $\mathbf{Co}$ controllability matrix is equal to 4, namely $rank(\mathbf{Co}) = 4 \equiv n$, i.e. the reference LTI system is controllable. Therefore, it is possible to design a reference states feedback controller $\mathbf{K}_{ref}$ so that $\mathbf{u}(t) = -\mathbf{K}_{ref}\mathbf{x}(t)$.
The $\mathbf{K}_{ref}$ optimal gain (for LQ optimal state feedback controller) can be designed by using the MATLAB$^{\text{TM}}$ *care* command. The design parameters in this given case are assumed to be: $\mathbf{Q} = \text{diag}(20, 10, 10, 10)$ and $\mathbf{R} = \text{diag}(1/20, 1/5, 1/5, 1/5)$, which provide fast poles and higher feedback gain at the critical state $x_1(t)$. The *care* command does calculate $\mathbf{X}$ as the unique solution of the control algebraic Ricatti equation (in the continuous-time domain) [27] as follows

$$\mathbf{A}^\top\mathbf{X}\mathbf{E} + \mathbf{E}^\top\mathbf{X}\mathbf{A} - (\mathbf{E}^\top\mathbf{X}\mathbf{B} + \mathbf{S})\mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{X}\mathbf{E} + \mathbf{S}^\top) + \mathbf{Q} = \mathbf{O} \ , \quad (32)$$

where the calculated optimal gain – besides $\mathbf{S} = \mathbf{0}$ and $\mathbf{E} = \mathbf{I}$ – is equal to $\mathbf{K}_{ref} = \mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{X}\mathbf{E} + \mathbf{S}^\top)$. The obtained optimal gain for the reference LTI system is:

$$\mathbf{K}_{ref} = \begin{bmatrix} -19.7359 & 0.4638 & 0.6369 & -0.9001\\ -0.1159 & 6.2073 & 1.0364 & 0.0073\\ -0.1592 & 1.0364 & 5.8609 & 0.0101\\ -0.2250 & -0.0073 & -0.0101 & -6.1272 \end{bmatrix} . \quad (33)$$

The $\mathbf{K}_{ref}$ feedback gain does provide that the eigenvalues of the closed loop become $\lambda_{ref,closed} = [-19.9798, -7.1725, -7.3062+0.1332i, -7.3062-0.1332i]^\top$ via $\mathbf{A}(\mathbf{p}_{ref}) - \mathbf{B}\mathbf{K}_{ref}$. The higher negative real parts of the eigenvalues provide fast transient part and stability, moreover, the occurring small complex parts do not cause high transient excursion.
Since $\mathbf{B}$ is invertible, the (9) can be used directly to calculate $\mathbf{K}(t)$ and realize the complementary LPV controller structure. Through the developed control law the structure of the complementary controller does provide that the strict equality of (7) will be satisfied over the operation. Namely, the $\lambda_{LPV,closed} = \lambda_{ref,closed}$ $\forall t(t \geq 0)$ everywhere in the state space (and parameter space) regardless the actual value of $\mathbf{p}(t)$ parameter vector.

3.3 Design of the Complementary LPV Observer

The design of the complementary LPV observer is similar to the previous controller case and the calculation steps follow the same straightforward path.
First, the observability of the reference LTI system has to be investigated as the critical point of the design. The rank of the observability matrix determines whether the system is observable or not. In this particular case, the rank of the $\mathbf{Ob}$ observability matrix $rank(\mathbf{Ob}) = 4 \equiv n$, i.e. the reference LTI system is observable.
The $\mathbf{G}_{ref}$ reference observer gain can be calculated by using the MATLAB's *place* command [27]. In practice, the eigenvalues of $\mathbf{A}-\mathbf{LC}$ should have more negative real parts (should be lower) than the system matrix in the $\lambda_{ref,closed}$ closed loop the eigenvalues in order to reach good observer dynamics (fast response and adaptivity). Consider, that $\lambda_{obs} = [-41, -43, -45, -47]\top$, where $\lambda_{obs,i} > \lambda_{ref,closed,i}$ $i = [1, 2, 3, 4]$.
The resulting $\mathbf{G}_{ref}$ becomes:

$$\mathbf{G}_{ref} = 10^3 \begin{bmatrix} 0.0298 & -0.0000 & 0 \\ -0.2951 & 1.9354 & 0 \\ -0.0071 & 0.0875 & 0 \\ 0.0007 & 0 & 0.0400 \end{bmatrix} . \tag{34}$$

Since, $\mathbf{C}$ is not invertible and (12) cannot be used directly, we have to apply the design process from Sec. 2.6.2. In this case $\mathbf{G}(t)$ can be calculated based on (21) and (22).

$$\mathbf{A}_{ref} - \mathbf{A}(\mathbf{p}(t)) = \mathbf{G}(t)\mathbf{C} = \sum_{i=1}^{2} \mathbf{g}(\mathbf{p}(t))_i \mathbf{c}_i^\top \rightarrow \mathbf{G}(t)$$
$$\mathbf{G}(t) = \begin{bmatrix} \frac{\Delta p_1(t)}{c_1} & 0 & 0 \\ \frac{\Delta p_2(t)}{c_1} & \frac{\Delta p_3(t)}{c_2} & 0 \\ \frac{\Delta p_4(t)}{c_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{-0.3333-p_1(t)}{c_1} & 0 & 0 \\ \frac{0-p_2(t)}{c_1} & \frac{-1-p_3(t)}{c_2} & 0 \\ \frac{-0.6667-p_4(t)}{c_1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{35}$$

The $rank(\mathbf{A}_{ref} - \mathbf{A}(t)) = 2$ which is equal to the number of dyads in the minimal dyadic structure. After calculating $\mathbf{G}(t)$ the (12) can be used to realize the complementary observer structure and because of the above described similarity the eigenvalue equality will be satisfied.The last missing piece is the feed forward compensation from Sec. 2.7. By using (24) the $\mathbf{N}_x(\mathbf{p}(t))$ and $\mathbf{N}_u(\mathbf{p}(t))$ can be calculated continuously during operation.
In this way the final control loop is realizable in accordance with Fig. 1.

3.4 Results

In this section the results of the simulations are detailed. Since the aim of the complementary LPV controller and observer is to enforce a particular LPV system and through the original nonlinear system to behave as the given LTI reference system, the focus during the presentation of the results is to highlight this property. In order to do that – beside keeping in mind the constraints – the corresponding signals of the nonlinear and LTI reference system will be presented and compared to each other. The simulations are carried out with the following system models:

1. Reference LTI system $\mathbf{S}_{ref}$: state vector $\mathbf{x}_{LTI}(t)$, observed reference state vector $\hat{\mathbf{x}}_{LTI}(t)$, output vector $\mathbf{y}_{LTI}(t)$, the permanent parameter vector $\mathbf{p}_{ref}$.

2. Original nonlinear system with complementary LPV controller and observer: state vector $\mathbf{x}_{orig}(t)$, observed state vector $\hat{\mathbf{x}}_{LPV}(t)$ coming from the complementary LPV observer (12), output vector of the nonlinear system $\mathbf{y}_{orig}(t)$, parameter vector $\mathbf{p}_{LPV}(t)$ generated by the observed states $\hat{\mathbf{x}}_{LPV}(t)$.

During the simulation the same settings were used in every case. The reference signal for the system states is the mentioned favorable steady values $\mathbf{r} = \mathbf{x}_{favorable} = [0, x_2^*, (2x_2^*/3), 0]^\top = [0, 2, 1.3333, 0]^\top$, hence the desired steady-state of the system to be $\mathbf{x}_\infty = \mathbf{r}$. The corresponding steady-state output is $\mathbf{y} = [0, 1.333, 0]^\top$. The initial state vector for every system was $\mathbf{x}(0) = [1.5, 3, 2, 0]^\top$. The initial state vector for the observers was equal to the desired steady-state values (since, this is known and determined) $\hat{\mathbf{x}}(0) = \mathbf{r}$. However, in this way there is an initial observation error, thus the dynamics of the observer can be analyzed. The simulation time was 1 time unit. This time span is enough to study the behavior of the system since all of the transients disappear under this time frame because of the fast operation and dynamics.
On Fig. 2. the variation of the states are presented over the simulated time span. Naturally, not all of the states are measurable ($x_1(t)$ and $x_2(t)$ cannot be measured directly). Albeit, – in order to reach a better understanding of the developed method – all of the corresponding states can be found on the diagram. The figure contains the following signals from the top to the bottom (started with the left column):

a) Vary of the $\mathbf{x}_{LTI}(t)$ states of the selected reference LTI system $\mathbf{S}_{ref}$ belongs to $\mathbf{p}_{ref}$

b) Vary of the $\mathbf{x}_{orig}(t)$ states of the original nonlinear time varying model

c) Comparison of the difference of the observed states based on the $\mathscr{L}_1(t)$ vector norm as follows: $\mathscr{L}_1(t) := \|\mathbf{x}_{LTI}(t) - \mathbf{x}_{orig}(t)\|_1$

d) Vary of the $\hat{\mathbf{x}}_{LTI}(t)$ observed states of the selected reference LTI system by the reference LTI observer

e) Vary of the $\hat{\mathbf{x}}_{LPV}(t)$ observed states of original nonlinear system coming from the complementary LPV observer

f) Comparison of the difference of the observed states based on the $\mathscr{L}_1(t)$ vector norm as follows: $\mathscr{L}_1(t) := \|\hat{\mathbf{x}}_{LTI}(t) - \hat{\mathbf{x}}_{LPV}(t)\|_1$

In practice, only the signals from Subfigs. d), e) and f) are available (accessible), since these are produced by the observers. However, the simulations can tell us useful information regarding the original states as well.

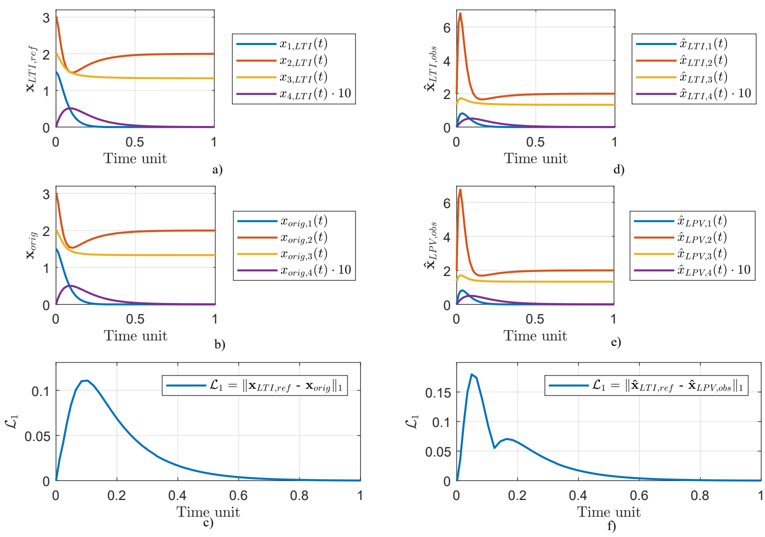

Figure 2
Comparison of the states during simulation. Due to the magnitude of x_4 was too low compared to other states it was multiplied by 10 to make it comparable, thus the order of magnitude became 10^1.

According to the simulations the developed complementary controller and observer structures performed well. Subfigs. a) and b) present the variation of the states of the reference LTI system ($\mathbf{S}_{ref}$) and the original nonlinear system, respectively. As it was stated, the $\mathbf{x}(0)$ initial values are the same in case of these systems. According to Subfig. c) there is a small deviation between the states based on the $\mathscr{L}_1(t)$ norm at the beginning which disappears over time. Furthermore, the magnitude of difference is small and can be neglected (since $\mathscr{L}_1(t) := \|\mathbf{x}_{LTI}(t) - \mathbf{x}_{orig}(t)\|_1$ which means there was only a small numerical difference between the states of the LTI reference system and the original nonlinear system).

Subfigs. d) and e) show that the same results regard to the reference observer and complementary LPV observer structure. The initial values of the observers were the same (and equal to the **r** reference). The Subfig. f) shows that the complementary observer structure performs well, thus the difference between the states of the observers were small and disappeared over time.
Finally, all of the states – with respect to the reference LTI system, the original nonlinear system, the reference LTI observer and the complementary LPV controlled system – reached the same final value what was the main target during operation regardless the variation of the parameter vector and the different initial conditions according to Fig. 2.

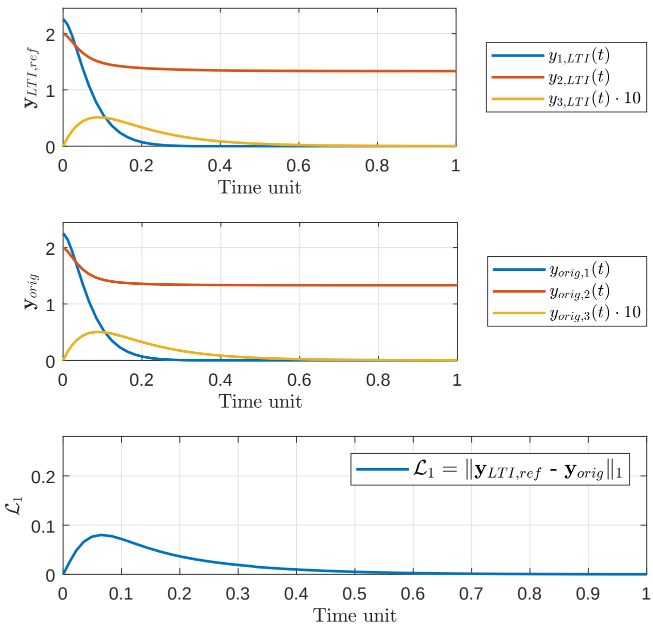

Figure 3
Comparison of the outputs during simulation. Due to the magnitude of y_3 was too low compared to other states it was multiplied by 10 to make it comparable, thus the order of magnitude became 10^1.

Figure 3 shows the variation of the output of the original nonlinear system ($\mathbf{y}_orig$) and the reference LTI system ($\mathbf{y}_{LTI,ref}$). As it can be seen the results correspond to the previous findings and the outputs behave as expected. At the beginning there is a small difference between the outputs (according to the defined $\mathscr{L}_1$) norm, but the deviation ceases over time. The results reflect that the complementary LPV controller and observer structure works well, thus it enforces that original nonlinear system to behave as the reference LTI system – and the numerical values of the outputs became equal over time as well.

The variation of the $\mathbf{p}(t)$ parameter vector converted from the observed states can be seen on Fig. 4. The figure strengthened the previous results, namely, regardless the variation of the parameter vector the completed LPV controller and observer structure is able to enforce the original nonlinear system to behave as the reference LTI system.

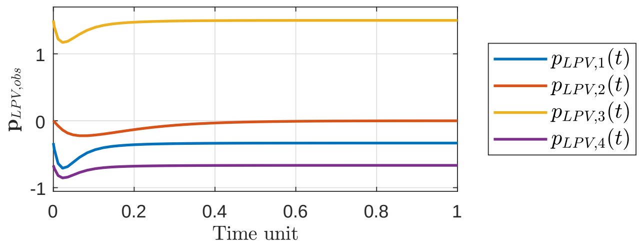

Figure 4
Vary of the parameter vector during simulation.

Conclusions

In this paper we presented a novel complementary LPV controller and observer design approach. The proposed method combines the classical state feedback with matrix similarity theorems, respectively. We analyzed the drawbacks, limitations and benefits of the introduced method.
The main advantages of this method is that it is able to provide appropriate, stable LPV controller and observer for the whole parameter domain by using a given reference LTI system as basis. Through the completed LPV controller and observer structure it is possible to enforce the nonlinear system to behave as the given LTI reference system.
We provided a practical example, namely, control of innate immune response. The results were satisfying since the completed LPV controller and observer structure was able to provide good control action and during operation the states of the reference LTI system and the original nonlinear system behaved similarly.
In our future work we are going to investigate the further generalization possibilities of the proposed techniques and we will try the methods in case of physical systems as well.

Acknowledgement

Gy. Eigner was supported by the ÚNKP-16-3/IV. New National Excellence Program of the Ministry of Human Capacities.
The author also thanks the support of the Robotics Special College of Óbuda University and the Applied Informatics and Applied Mathematics Doctoral School of Óbuda University.

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